On a mean-field model of interacting neurons

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Introduction

Model of coupled noisy Integrate and Fire neurons.
Mean-Field description through a **McKean-Vlasov SDE**.

From a **Dynamical System** point of view:
What are the **invariant measures** (equilibrium points for ODEs), what can we say about their **stability** (local, global) ? What happens if the invariant measure is not locally stable (**bifurcations**) ?

The model: Interspikes dynamics

N neurons characterized by their membrane potential:

$$V_t^i \in \mathbb{R}_+$$

Between the spikes, $(V_t^i)_{t \geq 0}$ solves a simple deterministic ODE:

$$\frac{dV_t^i}{dt} = b(V_t^i).$$

(Example: $b \equiv$ constant: the potential of each neuron grows linearly between its spikes).

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The model: Spiking dynamic

Each neuron i spikes randomly at a rate $f(V_t^i)$.

When such a spike occurs (say at time τ):

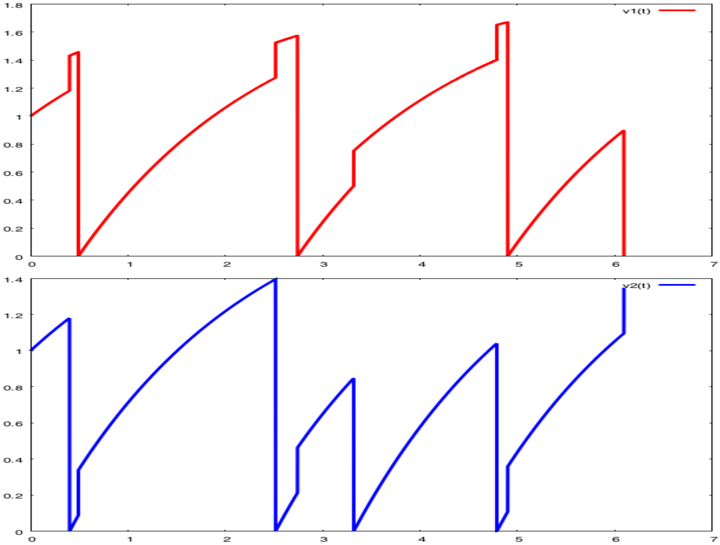
1. The potential of the neuron i is reset to 0:

$$V_\tau^i = 0$$

2. The potentials of the other neurons are increased by $J^{i \to j}$:

$$j \neq i, \; V_\tau^j = V_{\tau-}^j + J^{i \to j}.$$

Illustration with $N = 2$ neurons



The parameters of the problem

The 4 parameters of the model are:

1. the drift $b : \mathbb{R}_+ \to \mathbb{R}$, with $b(0) > 0$: it gives the dynamic of the neurons between the spikes
2. the rate function $f : \mathbb{R}_+ \to \mathbb{R}_+$: it encodes the probability for a neuron of a given potential to spike between t and $t + dt$.
3. The connectivity parameters $(J^{i \to j})_{i,j}$.
4. the law of the initial potentials: we assume the neurons are initially i.i.d. with probability law ν.

The particle systems

Let $(\mathbf{N}^i(du,dz))_{i=1,\dots,N}$ N independent Poisson measures on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity measure $dudz$.

Let $(V_0^i)_{i=1,\dots,N}$ a family of N random variables on $\mathbb{R}_+$, *i.i.d.* of law ν

Then (V_t^i) is a *càdlàg* process solution of the SDE:

$$\begin{cases} V_t^i = V_0^i + \int_0^t b(V_u^i)du + \sum_{j\neq i} J^{j\to i} \int_0^t \int_{\mathbb{R}_+} \mathbb{1}_{\{z \le f(V_{u-}^j)\}} \mathbf{N}^j(du,dz) \\ \qquad - \int_0^t \int_{\mathbb{R}_+} V_{u-}^i \mathbb{1}_{\{z \le f(V_{u-}^i)\}} \mathbf{N}^i(du,dz). \end{cases}$$

The limit equation

Simplification: $J^{i\to j} = \frac{J}{N}$ for some constant $J \geq 0$

$$V_t^i = V_0^i + \int_0^t b(V_u^i)du + \frac{J}{N}\sum_{j\neq i}\int_0^t \int_{\mathbb{R}_+} \mathbb{1}_{\{z\leq f(V_{u-}^j)\}}\mathbf{N}^j(du,dz) - \int_0^t \int_{\mathbb{R}_+} V_{u-}^i \mathbb{1}_{\{z\leq f(V_{u-}^i)\}}\mathbf{N}^i(du,dz).$$

$N \to \infty$: the **Mean-Field** equation

$$V_t = V_0 + \int_0^t b(V_u)du + J\int_0^t \mathbb{E}\, f(V_u)du - \int_0^t \int_{\mathbb{R}_+} V_{u-}\mathbb{1}_{\{z\leq f(V_{u-})\}}\mathbf{N}(du,dz) \qquad \text{(M-F)}$$

or equivalently:

$$\begin{cases} \frac{d}{dt}V_t = b(V_t) + J\,\mathbb{E}\, f(V_t) \\ + (V_t)_{t\geq 0} \text{ jumps to 0 with rate } f(V_t) \end{cases}$$

The Fokker-Planck PDE

The law of V_t solves (weakly) the Fokker-Planck equation:

$$\begin{cases} \frac{\partial}{\partial t}\nu(t,x) = -\frac{\partial}{\partial x}[(b(x)+Jr_t)\nu(t,x)] - f(x)\nu(t,x) \\ \nu(t,0) = \frac{r_t}{b(0)+Jr_t}, \quad r_t = \int_0^\infty f(x)\nu(t,x)dx. \end{cases}$$

N-L transport equation with a (N-L) boundary condition.

A brief tour of previous results

1. Many earlier considerations by physicists (**Keywords: hazard rate model, generalized I & F**)
2. A. De Masi, A. Galves, E. Löcherbach, E. Presutti, “Hydrodynamic limit for interacting neuron”
3. N. Fournier, E. Löcherbach, “On a toy model of interacting neurons” **results on the long time behavior for** $b \equiv 0$
4. A. Drogoul, R. Veltz “Hopf bifurcation in a nonlocal nonlinear transport equation stemming from stochastic neural dynamics” **numerical evidence of an Hopf bifurcation** in a closed setting.
5. A. Drogoul, R. Veltz “Exponential stability of the stationary distribution of a mean field of spiking neural network” **results on the long time behavior for** $b \equiv 0$.

Assumptions

Given $(a_t)_{t\geq 0} \in \mathcal{C}(\mathbb{R}_+, \mathbb{R})$ "any external current", let $\varphi^{(a.)}_{t,s}(x)$ be the flow solution of:

$$\frac{d}{dt}\varphi^{(a.)}_{t,s}(x) = b(\varphi^{(a.)}_{t,s}(x)) + a_t, \quad \varphi^{(a.)}_{s,s}(x) = x.$$

A1 $b : \mathbb{R}_+ \to \mathbb{R}$ is continuous, $b(0) > 0$, bounded from above.

A2 There exists a constant C: for all $(a_t)_{t\geq 0}, (d_t)_{t\geq 0}$:

$$\forall x \geq 0,\ \forall t \geq s,\ |\varphi^{(a.)}_{t,s}(x) - \varphi^{(d.)}_{t,s}(x)| \leq C \int_s^t |a_u - d_u| du.$$

A3 $f : \mathbb{R}_+ \to \mathbb{R}_+$ is $\mathcal{C}^1$ convex increasing, $f(0) = 0$ + some technical assumptions on the grow of f.

A4 The initial condition $\nu = \mathcal{L}(V_0)$ satisfies:

$$\nu(f^2) := \int_{\mathbb{R}_+} f^2(x)\nu(dx) < \infty.$$

What are the invariant measures of this N-L process?

$$V_t = V_0 + \int_0^t b(V_u)du + J\int_0^t \mathbb{E}\, f(V_u)du - \int_0^t \int_{\mathbb{R}_+} V_{u-}\mathbb{1}_{\{z \le f(V_{u-})\}}\mathbf{N}(du,dz)$$

In (M-F) replace the interactions $J\,\mathbb{E}\, f(X_t)$ by the constant $\alpha \geq 0$:

$$Y_t^\alpha = Y_0^\alpha + \int_0^t b(Y_u^\alpha)du + \alpha t - \int_0^t \int_{\mathbb{R}_+} Y_{u-}^\alpha \mathbb{1}_{\{z \le f(Y_{u-}^\alpha)\}}\mathbf{N}(du,dz).$$

This process has an unique invariant measure given by:

$$\nu_\alpha^\infty(dx) = \frac{\gamma(\alpha)}{b(x)+\alpha}\exp\left(-\int_0^x \frac{f(y)}{b(y)+\alpha}dy\right)\mathbb{1}_{\{x\in[0,\sigma_\alpha]\}}dx,$$

- $\sigma_\alpha = \lim_{t\to\infty}\varphi_{t,0}^\alpha(0) \in \mathbb{R}_+^* \cup \{+\infty\}$.
- $\gamma(\alpha)$ is the normalizing factor (such that $\int \nu_\alpha^\infty(dx) = 1$.)
- It holds that $\gamma(\alpha) = \nu_\alpha^\infty(f)$.

The invariant measures of (M-F) are exactly: $\{\nu_\alpha^\infty : \alpha = J\gamma(\alpha),\ \alpha \geq 0\}$.

The case of small interactions

Theorem (C., Tanré, Veltz 2018)

*Under **A1, A2, A3, A4**:*

1. *the N-L SDE* (M-F) *has a path-wise unique solution with* $\sup_{t \geq 0} \mathbb{E} f(V_t) < \infty$.
2. *if the interaction parameter J is small enough, then (V_t) has an **unique invariant measure** which is **globally stable**: starting from any initial condition, V_t converges in law to the unique invariant measure. The convergence is exponentially fast.*

Examples

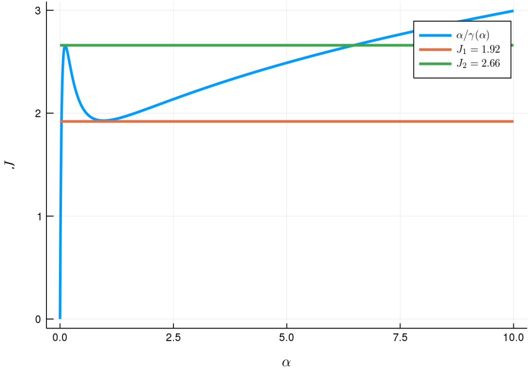
Consider for all $x \geq 0$:

$$b(x) = b_0 - b_1 x, \quad f(x) = x^{\xi}.$$

For $b_0 > 0, b_1 \geq 0$ and $\xi \geq 1$, its satisfies all the assumptions.

Examples

For $b(x) = 0.1 - x$, $f(x) = x^2$.



- $J < J_1$: one unique invariant measure.
- $J_1 < J < J_2$: **three** invariant measures, 2 are stable: **bi-stability**.
- $J > J_2$: one unique invariant measure.

Examples

For $b(x) = 2 - 2x$, $f(x) = x^{10}$.

Always exactly one invariant measure. But if $J \in [0.73, 1.04]$ spontaneous oscillation of $t \to \mathbb{E}\, f(V_t)$ appears! **The law of V_t asymptotically oscillates** (Video !).

The invariant measure looses its stability. There is a **Hopf bifurcation** for $J \approx 0.73$.

Sketch of the Proof

1) Introduce a **linearized version** of the N-L equation (M-F).
Given $(a_t)_{t\geq 0}$, consider

$$Y_t^{\nu,(a.)} = Y_0^{\nu,(a.)} + \int_0^t b(Y_u^{\nu,(a.)})du + \int_0^t a_u du - \int_0^t \int_{\mathbb{R}_+} Y_{u-}^{\nu,(a.)} \mathbb{1}_{\{z \leq f(Y_{u-}^{\nu,(a.)})\}} \mathbf{N}(du, dz).$$

The interactions $J\,\mathbb{E}\,f(V_t)$ have been replaced by a_t.
Then $(Y_t^{\nu,(a.)})$ is a solution of (M-F) if and only if:

$$\forall t \geq 0 : \; a_t = J\,\mathbb{E}\,f(Y_t^{\nu,(a.)}).$$

Sketch of the Proof

2) The jump rate of this linearized process solves a **Volterra equation**: let $r^{\nu}_{(a.)}(t) := \mathbb{E}\, f(Y_t^{\nu,(a.)})$. Then

$$\forall t \geq 0,\ r^{\nu}_{(a.)}(t) = K^{\nu}_{(a.)}(t,0) + \int_0^t K^{\delta_0}_{(a.)}(t,u) r^{\nu}_{(a.)}(u) du,$$

with for all $x \geq 0, t \geq s$

$$K^{\delta_x}_{(a.)}(t,s) := f(\varphi^{(a.)}_{t,s}(x)) \exp\left(- \int_s^t f(\varphi^{(a.)}_{u,s}(x)) du \right),$$

$$K^{\nu}_{(a.)}(t,s) := \int_0^{\infty} K^{\delta_x}_{(a.)}(t,s) \nu(dx).$$

Sketch of the Proof

3) We first study the case $(a_\cdot)$ constant and equal to α.

In that case the Volterra equation become a **convolution Volterra equation.**

We prove that for all $0 \le \lambda < \lambda^*_\alpha$

$$\sup_{t \ge 0} |\,\mathbb{E}\, f(Y^\alpha_t) - \gamma(\alpha)| e^{\lambda t} < \infty.$$

The number $\lambda^*_\alpha > 0$ is the largest real part of the Complex zeros of the Laplace transform of $H_\alpha(t) := \exp\left(-\int_0^t f(\varphi^\alpha_u) du\right)$.

Sketch of the Proof

4) **Main difficulty.** Using a perturbation method, we prove that for any current (a_t) :

$$\text{If } \sup_{t \geq 0} |a_t - \alpha| e^{\lambda t} < \infty, \qquad \text{for some } 0 < \lambda < \lambda^*_\alpha$$

$$\text{Then } \sup_{t \geq 0} |\,\mathbb{E}\, f(Y_t^{\nu,(a.)}) - \gamma(\alpha)| e^{\lambda t} < \infty.$$

The speed of convergence is the same !

Sketch of the Proof

5) We conclude using a Picard Iteration scheme that $\mathbb{E} f(V_t)$ converges at an exponential speed to $\gamma(\alpha)$.

We consider the following Picard Iteration:

$$a_{n+1}(t) = J\,\mathbb{E}\, f(Y_t^{\nu,(a_n.)}), \quad a_0 = \alpha$$

It holds that $\sup_{t\geq 0} |a_n(t) - \alpha| e^{\lambda t} < \infty$. The condition J **is small enough** ensures that the constants does not explode with n.
We deduce that $\sup_{t\geq 0} |J\,\mathbb{E}\, f(V_t) - \alpha| e^{\lambda t} < \infty$, provided that $\lambda < \lambda_\alpha^*$.

It is then not hard to conclude that V_t converges in law to ν_α^∞ at an exponential speed.

Non-linear local stability

What happens for larger weights J?

Definition

Equip $\mathcal{M}(f^2)$ with $d(\nu,\mu) = \int_{\mathbb{R}} [1 + f(x)] |\nu - \mu|(dx)$.
Let ν_α^∞ be an invariant measure of (M-F). We say it is **locally stable** if there exists some $\epsilon > 0$ and $C, \lambda > 0$ such that:

$$\forall \nu \in \mathcal{M}(f^2),\ d(\nu, \nu_\alpha^\infty) < \epsilon \implies d(\nu_t, \nu_\alpha^\infty) \leq C e^{-\lambda t},$$

with $\nu_t = \mathcal{L}(V_t)$.

Non-linear local stability

Our key tool to study the stability is to look at the zeros of

$$\begin{array}{rccl} \Phi_\alpha : & \mathcal{M}(f^2) \times L^\infty_\lambda & \to & L^\infty_\lambda \\ & (\nu,\ h) & \mapsto & (\alpha + h) - J r^\nu_{\alpha+h} \end{array}$$

Here $L^\infty_\lambda = \{x : \mathbb{R}_+ \to \mathbb{R} \mid ||x||^\infty_\lambda < \infty\}$ with $||x||^\infty_\lambda = \text{esssup}_{t\geq 0}|x(t)|e^{\lambda t}$.

Lemma

The function Φ_α is continuous with respect to ν and C^1 with respect to h. Moreover, there exists a function Θ_α such that $\forall 0 < \lambda < \lambda_\alpha,\ \Theta_\alpha \in L^1_\lambda$ and

$$\forall c \in L^\infty_\lambda,\ D_h\Phi(\nu^\infty_\alpha, 0) \cdot c = c - \Theta_\alpha * c.$$

The function Θ_α is known explicitly in term of f, b and α.

Non-linear local stability

Let $\widehat{\Theta}_\alpha(z)$ the Laplace transform of Θ_α.

Theorem

Assume all the complex roots of $z \mapsto \widehat{\Theta}_\alpha(z) - 1$ are located on the left half-plane. Then the invariant measure ν_α^∞ is locally stable.

Examples for which this theorem applies. For J small enough it is always satisfied. If $b \equiv 0$, it is satisfied (and the invariant measure is always locally stable, whatever the value of the weight J). When this theorem **does not** apply (in some way), spontaneous oscillations may exists through an Hopf bifurcation...

Conclusion and perspectives

- The mean-field equation is a McKean-Vlasov SDE and we study its long time behavior.
- Small connectivity (J small enough) $\implies$ relaxation to equilibrium.
- The model can be summarized by a neat **non-linear Volterra Integral equation**.
- It is possible to study finely the **local stability** of an invariant measure and to predict **Hopf bifurcation** (Work in Progress!).
- There is a straightforward extension to multi-populations, including **excitatory and inhibitory**.

The paper : arXiv:1810.08562
"Long time behavior of a mean-field model of interacting neurons"