On a mean-field model of interacting neurons

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Introduction

Model of coupled noisy Integrate and Fire neurons.
Mean-Field description through a **McKean-Vlasov SDE**.

From a **Dynamical System** point of view:
What are the **invariant measures** (equilibrium points for ODEs), what can we say about their **stability** (local, global) ? What happens if the invariant measure is not locally stable (**bifurcations**) ?

The model: Interspikes dynamics

N neurons characterized by their membrane potential:

$$V_t^i \in \mathbb{R}_+$$

Between the spikes, $(V_t^i)_{t\geq 0}$ solves a simple deterministic ODE:

$$\frac{dV_t^i}{dt} = b(V_t^i).$$

(Example: $b \equiv$ constant: the potential of each neuron grows linearly between its spikes).

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The model: Spiking dynamic

Each neuron i spikes randomly at a rate $f(V_t^i)$.

When such a spike occurs (say at time τ):

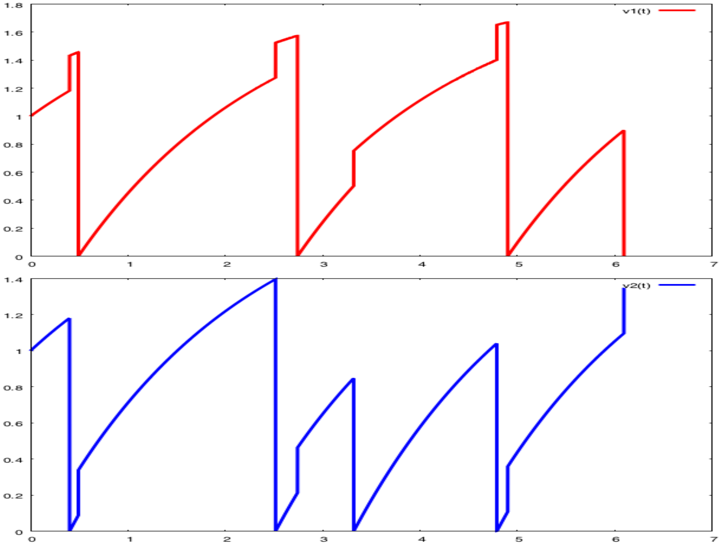
1. The potential of the neuron i is reset to 0:

$$V_\tau^i = 0$$

2. The potentials of the other neurons are increased by $J^{i \to j}$:

$$j \neq i, \; V_\tau^j = V_{\tau-}^j + J^{i \to j}.$$

Illustration with $N = 2$ neurons



The parameters of the problem

The 4 parameters of the model are:

1. the drift $b : \mathbb{R}_+ \to \mathbb{R}$, with $b(0) > 0$: it gives the dynamic of the neurons between the spikes
2. the rate function $f : \mathbb{R}_+ \to \mathbb{R}_+$: it encodes the probability for a neuron of a given potential to spike between t and $t + dt$.
3. The connectivity parameters $(J^{i \to j})_{i,j}$.
4. the law of the initial potentials: we assume the neurons are initially i.i.d. with probability law ν.

A choice of f.

$$f(x) = (\frac{x}{\vartheta})^{\xi},$$

with: $\xi \geq 1$ (large), $\vartheta > 0$.

When $V_t^i \gg \vartheta$, $f(V_t^i) \gg 1 \implies$ large probability to spike between t and $t + dt$.

When $V_t^i \ll \vartheta$, $f(V_t^i) \ll 1 \implies$ low probability to spike between t and $t + dt$.

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The particle systems

Let $(\mathbf{N}^i(du,dz))_{i=1,\dots,N}$ N independent Poisson measures on $\mathbb{R}_+\times\mathbb{R}_+$ with intensity measure $dudz$.

Let $(V_0^i)_{i=1,\dots,N}$ a family of N random variables on $\mathbb{R}_+$, *i.i.d.* of law ν

Then (V_t^i) is a *càdlàg* process solution of the SDE:

$$\begin{cases} V_t^i = V_0^i + \displaystyle\int_0^t b(V_u^i)du + \sum_{j\neq i} J^{j\to i}\int_0^t\int_{\mathbb{R}_+} \mathbb{1}_{\{z\le f(V_{u-}^j)\}}\mathbf{N}^j(du,dz) \\ \qquad - \displaystyle\int_0^t\int_{\mathbb{R}_+} V_{u-}^i \mathbb{1}_{\{z\le f(V_{u-}^i)\}}\mathbf{N}^i(du,dz). \end{cases}$$

The limit equation

Simplification: $J^{i \to j} = \frac{J}{N}$ for some constant $J \geq 0$

$$V_t^i = V_0^i + \int_0^t b(V_u^i)du + \frac{J}{N}\sum_{j \neq i}\int_0^t \int_{\mathbb{R}_+} \mathbb{1}_{\{z \leq f(V_{u-}^j)\}} \mathbf{N}^j(du, dz) - \int_0^t \int_{\mathbb{R}_+} V_{u-}^i \mathbb{1}_{\{z \leq f(V_{u-}^i)\}} \mathbf{N}^i(du, dz).$$

$N \to \infty$: the **Mean-Field** equation

$$V_t = V_0 + \int_0^t b(V_u)du + J\int_0^t \mathbb{E}\, f(V_u)du - \int_0^t \int_{\mathbb{R}_+} V_{u-} \mathbb{1}_{\{z \leq f(V_{u-})\}} \mathbf{N}(du, dz) \tag{M-F}$$

or equivalently:

$$\begin{cases} \dfrac{d}{dt} V_t = b(V_t) + J\,\mathbb{E}\, f(V_t) \\ \\ + \ (V_t)_{t \geq 0} \text{ jumps to 0 with rate } f(V_t) \end{cases}$$

The Fokker-Planck PDE

The law of V_t solves (weakly) the Fokker-Planck equation:

$$\begin{cases} \dfrac{\partial}{\partial t}\nu(t,x) = -\dfrac{\partial}{\partial x}[(b(x)+Jr_t)\nu(t,x)] - f(x)\nu(t,x) \\ \nu(t,0) = \dfrac{r_t}{b(0)+Jr_t}, \quad r_t = \displaystyle\int_0^\infty f(x)\nu(t,x)dx. \end{cases}$$

N-L transport equation with a (N-L) boundary condition.

A brief tour of previous results

1. Many earlier considerations by physicists.
2. A. De Masi, A. Galves, E. Löcherbach, E. Presutti, “Hydrodynamic limit for interacting neuron”
3. N. Fournier, E. Löcherbach, “On a toy model of interacting neurons” **results on the long time behavior for** $b \equiv 0$
4. A. Drogoul, R. Veltz “Hopf bifurcation in a nonlocal nonlinear transport equation stemming from stochastic neural dynamics” **numerical evidence of an Hopf bifurcation** in a closed setting.
5. A. Drogoul, R. Veltz “Exponential stability of the stationary distribution of a mean field of spiking neural network” **results on the long time behavior for** $b \equiv 0$.

Assumptions

Given $(a_t)_{t\geq 0} \in \mathcal{C}(\mathbb{R}_+, \mathbb{R})$ "any external current", let $\varphi^{(a.)}_{t,s}(x)$ be the flow solution of:

$$\frac{d}{dt}\varphi^{(a.)}_{t,s}(x) = b(\varphi^{(a.)}_{t,s}(x)) + a_t, \quad \varphi^{(a.)}_{s,s}(x) = x.$$

A1 $b : \mathbb{R}_+ \to \mathbb{R}$ is continuous, $b(0) > 0$, bounded from above.

A2 There exists a constant C: for all $(a_t)_{t\geq 0}, (d_t)_{t\geq 0}$:

$$\forall x \geq 0,\ \forall t \geq s,\ |\varphi^{(a.)}_{t,s}(x) - \varphi^{(d.)}_{t,s}(x)| \leq C \int_s^t |a_u - d_u| du.$$

A3 $f : \mathbb{R}_+ \to \mathbb{R}_+$ is $\mathcal{C}^1$ convex increasing, $f(0) = 0$ + some technical assumptions on the grow of f.

A4 The initial condition $\nu = \mathcal{L}(V_0)$ satisfies:

$$\nu(f^2) := \int_{\mathbb{R}_+} f^2(x)\nu(dx) < \infty.$$

What are the invariant measures of this N-L process?

$$V_t = V_0 + \int_0^t b(V_u)du + J\int_0^t \mathbb{E}\, f(V_u)du - \int_0^t \int_{\mathbb{R}_+} V_{u-}\mathbb{1}_{\{z \le f(V_{u-})\}}\mathbf{N}(du, dz)$$

In (M-F) replace the interactions $J\,\mathbb{E}\, f(X_t)$ by the constant $\alpha \ge 0$:

$$Y_t^\alpha = Y_0^\alpha + \int_0^t b(Y_u^\alpha)du + \alpha t - \int_0^t \int_{\mathbb{R}_+} Y_{u-}^\alpha \mathbb{1}_{\{z \le f(Y_{u-}^\alpha)\}}\mathbf{N}(du, dz).$$

This process has an unique invariant measure given by:

$$\nu_\alpha^\infty(dx) = \frac{\gamma(\alpha)}{b(x)+\alpha}\exp\left(-\int_0^x \frac{f(y)}{b(y)+\alpha}dy\right)\mathbb{1}_{\{x\in[0,\sigma_\alpha]\}}dx,$$

- $\sigma_\alpha = \lim_{t\to\infty} \varphi_{t,0}^\alpha(0) \in \mathbb{R}_+^* \cup \{+\infty\}$.
- $\gamma(\alpha)$ is the normalizing factor (such that $\int \nu_\alpha^\infty(dx) = 1$.)
- It holds that $\gamma(\alpha) = \nu_\alpha^\infty(f)$.

The invariant measures of (M-F) are exactly: $\{\nu_\alpha^\infty : \alpha = J\gamma(\alpha),\ \alpha \ge 0\}$.

Main result

Theorem (C., Tanré, Veltz 2018)

*Under **A1, A2, A3, A4**:*

1. *the N-L SDE* (M-F) *has a path-wise unique solution with* $\sup_{t\geq 0} \mathbb{E}\, f(V_t) < \infty$.
2. *if the interaction parameter J is small enough, then (V_t) has an **unique invariant measure** which is **globally stable**: starting from any initial condition, V_t converges in law to the unique invariant measure. The convergence is exponentially fast.*

Examples

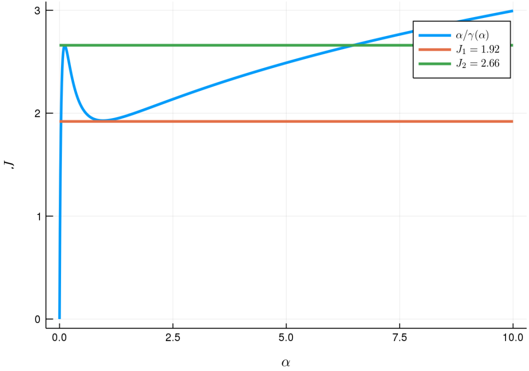
Consider for all $x \geq 0$:

$$b(x) = b_0 - b_1 x, \quad f(x) = x^{\xi}.$$

For $b_0 > 0, b_1 \geq 0$ and $\xi \geq 1$, its satisfies all the assumptions.

Examples

For $b(x) = 0.1 - x$, $f(x) = x^2$.



- $J < J_1$: one unique invariant measure.
- $J_1 < J < J_2$: **three** invariant measures, 2 are stable: **bi-stability**.
- $J > J_2$: one unique invariant measure.

Examples

For $b(x) = 2 - 2x$, $f(x) = x^{10}$.

Always exactly one invariant measure. But if $J \in [0.73, 1.04]$ spontaneous oscillation of $t \to \mathbb{E} f(V_t)$ appears! **The law of V_t asymptotically oscillates** (Video !).

The invariant measure looses its stability. There is a **Hopf bifurcation** for $J \approx 0.73$.

Sketch of the Proof

1) Introduce a **linearized version** of the N-L equation (M-F).
Given $(a_t)_{t\geq 0}$, consider

$$Y_t^{\nu,(a.)} = Y_0^{\nu,(a.)} + \int_0^t b(Y_u^{\nu,(a.)})du + \int_0^t a_u du - \int_0^t \int_{\mathbb{R}_+} Y_{u-}^{\nu,(a.)} \mathbb{1}_{\{z\leq f(Y_{u-}^{\nu,(a.)})\}} \mathbf{N}(du,dz).$$

The interactions $J\,\mathbb{E}\,f(V_t)$ have been replaced by a_t.
Then $(Y_t^{\nu,(a.)})$ is a solution of (M-F) if and only if:

$$\forall t \geq 0:\ \ a_t = J\,\mathbb{E}\,f(Y_t^{\nu,(a.)}).$$

Sketch of the Proof

2) The jump rate of this linearized process solves a **Volterra equation**: let $r^{\nu}_{(a.)}(t) := \mathbb{E}\, f(Y^{\nu,(a.)}_t)$. Then

$$\forall t \geq 0,\ r^{\nu}_{(a.)}(t) = K^{\nu}_{(a.)}(t,0) + \int_0^t K^{\delta_0}_{(a.)}(t,u) r^{\nu}_{(a.)}(u) du,$$

with for all $x \geq 0, t \geq s$

$$K^{\delta_x}_{(a.)}(t,s) := f(\varphi^{(a.)}_{t,s}(x)) \exp\left(- \int_s^t f(\varphi^{(a.)}_{u,s}(x)) du \right),$$

$$K^{\nu}_{(a.)}(t,s) := \int_0^{\infty} K^{\delta_x}_{(a.)}(t,s) \nu(dx).$$

Sketch of the Proof

3) We first study the case $(a.)$ constant and equal to α.

In that case the Volterra equation become a **convolution Volterra equation.**

We prove that for all $0 \leq \lambda < \lambda_\alpha^*$

$$\sup_{t \geq 0} |\,\mathbb{E}\, f(Y_t^\alpha) - \gamma(\alpha)| e^{\lambda t} < \infty.$$

The number $\lambda_\alpha^* > 0$ is the largest real part of the Complex zeros of the Laplace transform of $H_\alpha(t) := \exp\left(-\int_0^t f(\varphi_u^\alpha) du\right)$.

Sketch of the Proof

4) **Main difficulty.** Using a perturbation method, we prove that for any current (a_t) :

If $\sup_{t \geq 0} |a_t - \alpha| e^{\lambda t} < \infty$, for some $0 < \lambda < \lambda^*_\alpha$

Then $\sup_{t \geq 0} | \mathbb{E} f(Y_t^{\nu,(a.)}) - \gamma(\alpha)| e^{\lambda t} < \infty.$

The speed of convergence is the same !

Sketch of the Proof

5) We conclude using a Picard Iteration scheme that $\mathbb{E}\, f(V_t)$ converges at an exponential speed to $\gamma(\alpha)$.

We consider the following Picard Iteration:

$$a_{n+1}(t) = J\, \mathbb{E}\, f(Y_t^{\nu,(a_n.)}), \quad a_0 = \alpha$$

It holds that $\sup_{t\geq 0} |a_n(t) - \alpha| e^{\lambda t} < \infty$. The condition J **is small enough** ensures that the constants does not explode with n.
We deduce that $\sup_{t\geq 0} |J\, \mathbb{E}\, f(V_t) - \alpha| e^{\lambda t} < \infty$, provided that $\lambda < \lambda_\alpha^*$.

It is then not hard to conclude that V_t converges in law to ν_α^∞ at an exponential speed.

Conclusion and perspectives

- The mean-field equation is a McKean-Vlasov SDE and we study its long time behavior.
- Small connectivity (J small enough) $\implies$ relaxation to equilibrium.
- The model can be summarized by a neat **non-linear Volterra Integral equation**.
- It is possible to study finely the **local stability** of an invariant measure and to predict **Hopf bifurcation** (Work in Progress!).
- There is a straightforward extension to multi-populations, including **excitatory and inhibitory**.

The paper : arXiv:1810.08562

"Long time behavior of a mean-field model of interacting neurons"