A Mean-Field model of interacting neurons

QUENTIN CORMIER - ETIENNE TANRÉ - ROMAIN VELTZ

INRIA - QUENTIN.CORMIER@INRIA.FR

1. THE MODEL

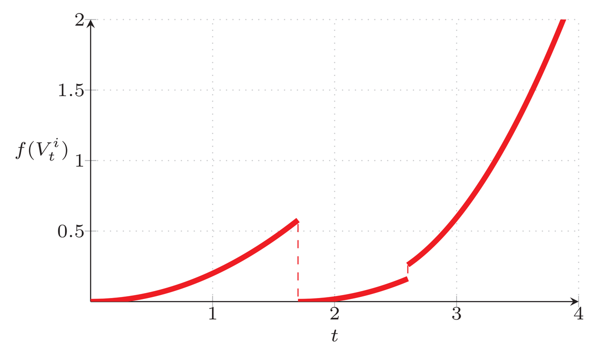
We model neurons in interactions. The model is known in the literature as the **generalized integrate-and-fire (GIF)** or as the **Escape noise model**.
Key results. In the **Mean-Field (M-F)** limit - where the number of neurons goes to infinity - we study the **long time behavior** of the network. We show that, depending on the average interaction strength, either the system can **stabilize to a steady states** or **oscillate indefinitely**. We develop specific mathematical tools to **classify the stability of the invariant measures** and **predict the emergence of spontaneous oscillations**.

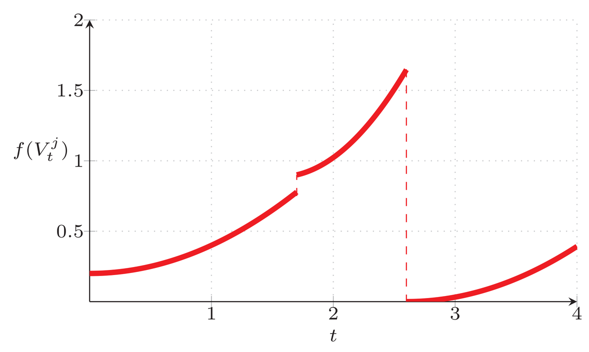
We consider $N \geq 1$ neurons, characterized by their membrane potential $(V^i_t)_{t\geq 0}$, $i \in 1, .., N$.
Between the spikes, $(V^i_t)_{t\geq 0}$ solves:

$$dV_t = b(V_t)dt$$

Neuron i **spike randomly** at time t with rate $f(V^i_t)$. Then

1. The potential of neuron i is reset to zero: $V^i_t = 0$
2. The other neurons $j \neq i$ receive a kick: $V^j_t = V^j_{t-} + J_{i\to j}$





The parameters. (a) The drift term $b : \mathbb{R}_+ \to \mathbb{R}$, it gives the sub-threshold dynamics. (b) The rate function $f : \mathbb{R}_+ \to \mathbb{R}_+$: $f(v)dt$ is the probability for a neuron with a potential v to spike between t and $t + dt$. (c) The connectivity matrix $(J_{i\to j})_{i,j}$ (deterministic and constant in time). (d) The initial conditions of the neuron.

A0. For all $x \in \mathbb{R}_+$, $b(x) = b_0 - \kappa x$ and $f(x) = (x_+)^p$, for some constants $b_0, \kappa \geq 0$ and $p \geq 1$.

2. THE MEAN-FIELD LIMIT

A1. Assume the initial conditions $V^1_0, \dots V^N_0$ are i.i.d. with probability law $\nu \in \mathcal{M}(f^2)$, that is: $\int_{\mathbb{R}} f^2(x)\nu(dx) < \infty$.
A2. Assume $J_{i\to j} = \frac{J}{N}$ for some constant $J \geq 0$.
As $N \to +\infty$, $\mathcal{L}((V^i_t)_{t\geq 0}) \to_N \mathcal{L}((V_t)_{t\geq 0})$ where $(V_t)_{t\geq 0}$ solves the **Mckean-Vlasov** equation

(1) $$\frac{d}{dt}V_t = b(V_t) + J\mathbb{E}\, f(V_t) \quad \text{and } V_t \text{ jumps to 0 with rate } f(V_t).$$

↪ $\mathbb{E}\, f(V_t)$ is the mean number of spikes per unit of time in the network.

Theorem 1 *The mean-field SDE (1) has a path-wise unique solution $(V_t)_{t\geq 0}$. Moreover, it holds that $\sup_{t\geq 0} \mathbb{E}\, f(V_t) < \infty$.*

Let $\nu(t, \cdot)$ be the law of V_t at time $t \geq 0$. It solves (weakly) the **Fokker-Planck PDE**:

$$\begin{cases} \frac{\partial}{\partial t}\nu(t,x) = -\frac{\partial}{\partial x}[(b(x) + Jr_t)\nu(t,x)] - f(x)\nu(t,x), \\ \nu(0,\cdot) = \nu, \quad r_t = \int_0^\infty f(x)\nu(t,x)dx, \\ \nu(t,0) = \frac{r_t}{b(0) + Jr_t}. \end{cases}$$

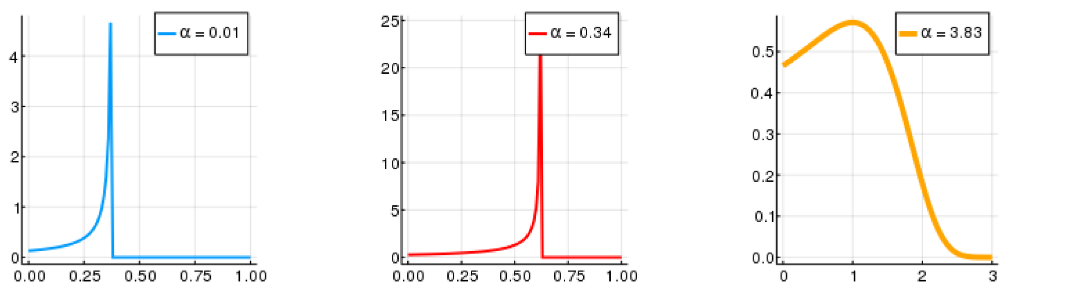
↪ $r_t = \mathbb{E}\, f(V_t)$; it is the key quantity to study to understand the model.

4. THE INVARIANT MEASURES

Theorem 3 *The invariant probability measures of the mean-field SDE (1) are $\{\nu^\infty_\alpha \mid \alpha = J\gamma(\alpha),\ \alpha \geq 0\}$, with*

$$\nu^\infty_\alpha(x)dx := \frac{\gamma(\alpha)}{\alpha + b(x)} \exp\left(-\int_0^x \frac{f(y)}{\alpha + b(y)}dy\right)\mathbb{1}_{x\in[0,\sigma_\alpha]}dx,$$

where σ_α is the limit of the deterministic flow of the ODE driven by $b(x) + \alpha$ and $\gamma(\alpha)$ is the normalizing factor. It holds that $\nu^\infty_\alpha(f) = \gamma(\alpha)$.



Example: $f(x) = x^3$, $b(x) = 0.28 - x$, $J = 2$: there are three invariant measures ($\alpha_1 \approx 0.01$, $\alpha_2 \approx 0.34$ and $\alpha_3 \approx 3.83$)

3. VOLTERRA EQUATION

The difficulty: there is no closed formula for $t \mapsto \mathbb{E}\, f(V_t)$.
The Ito formula gives

$$\frac{d}{dt}\mathbb{E}\, f(V_t) = \mathbb{E}\, f'(V_t)\,[b(V_t) + J\,\mathbb{E}\, f(V_t))] - \mathbb{E}\, f^2(V_t).$$

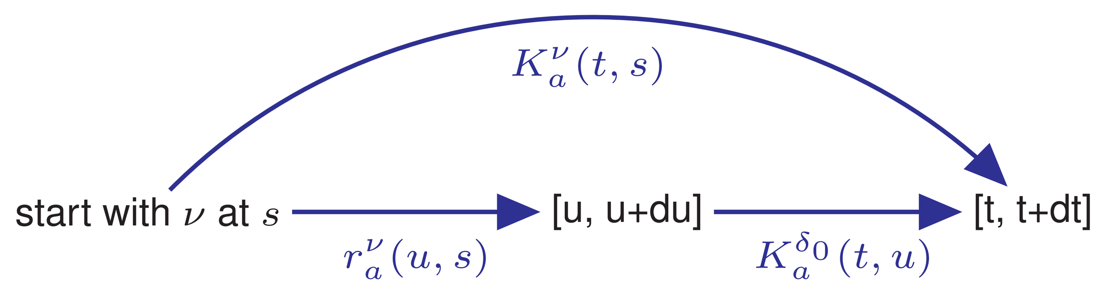
In particular $\frac{d}{dt}\mathbb{E}\, f(V_t)|_{t=0}$ depends on $\mathbb{E}\, f(V_0)$ but also on $\mathbb{E}\, f'(V_0)$ and on $\mathbb{E}\, f^2(V_0)$.
The linearized process. Given an "external current" $(a_t)_{t\geq 0} \in \mathcal{C}(\mathbb{R}_+)$ we define $Y^{\nu,a}_t$, starting at time s with law ν, solution of:

(2) $$\frac{d}{dt}Y^{\nu,a}_{t,s} = b(Y^{\nu,a}_{t,s}) + a_t \quad + \quad \text{jumps to 0 at rate } f(Y^{\nu,a}_{t,s})$$

The survival function and its density. Define

$$\tau^{\nu,a}_s := \text{time of the first jump of } Y^{\nu,a}_{t,s}, \quad r^\nu_a(t,s) = \mathbb{E}\, f(Y^{\nu,a}_{t,s})$$

$$H^\nu_a(t,s) := \mathbb{P}(\tau^{\nu,a}_s > t), \quad K^\nu_a(t,s) := -\frac{d}{dt}\mathbb{P}(\tau^{\nu,a}_s > t).$$



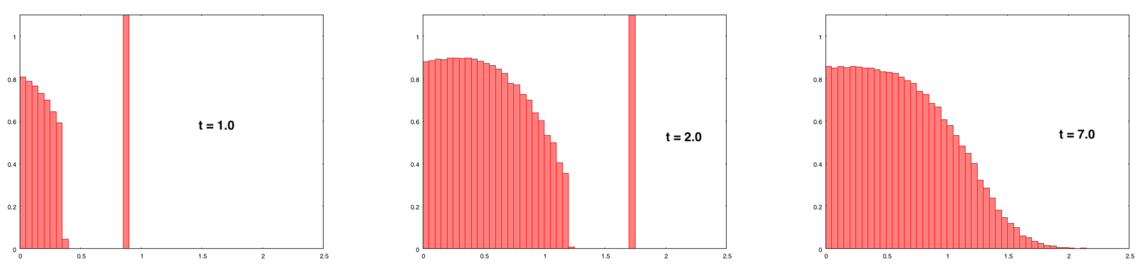
We have $r^\nu_a(t,s) = \lim_{\delta\downarrow 0}\frac{1}{\delta}\mathbb{P}(\text{there is a jump in } [t, t+\delta])$.

Proposition 1 *$r^\nu_a(t,s)$ is the solution of the non-homogenous Volterra equation:*

$$r^\nu_a(t,s) = K^\nu_a(t,s) + \int_s^t K^{\delta_0}(t,u)r^\nu_a(u,s)du.$$

↪ $a_t := Jr_t = J\,\mathbb{E}\, f(V_t)$ is the unique solution of $a = Jr^\nu_a$. When $a \equiv \alpha$ is constant, it reduces to a **linear convolution Volterra equation**. We used **Laplace transform** to study it. The asymptotic of $t \mapsto r^\nu_\alpha(t)$ is related to the location of the zeros of the Laplace transform of $t \mapsto H^{\delta_0}_\alpha(t)$.

Theorem 2 (See [3]) *Given b and f, one can find a weight $J_0 > 0$ s.t. for all $J \in [0, J_0]$, the mean-filed SDE has an unique invariant measure which is* ***globally stable****.*



Example: initial measure $\nu = \delta_{0.5}$, $J = 1.0$, $\mu = 0.1$: evolution of the law of V_t as t goes to infinity.

5. LOCAL STABILITY

What happens for larger weights J?
Def. Equip $\mathcal{M}(f^2)$ with $d(\nu, \mu) = \int_{\mathbb{R}}[1 + f(x)]|\nu - \mu|(dx)$.
Let ν^∞_α be an invariant measure of (1). We say it is **locally stable** if there exists some $\epsilon > 0$ and $C, \lambda > 0$ such that:

$$\forall \nu \in \mathcal{M}(f^2),\ d(\nu, \nu^\infty_\alpha) < \epsilon \implies d(\nu_t, \nu^\infty_\alpha) \leq Ce^{-\lambda t},$$

with $\nu_t = \mathcal{L}(V_t)$. Our key tool to study the stability is to look at the zeros of

$$\begin{array}{rccc} \Phi_\alpha : & \mathcal{M}(f^2) \times L^\infty_\lambda & \to & L^\infty_\lambda \\ & (\nu,\ h) & \mapsto & (\alpha + h) - Jr^\nu_{\alpha+h} \end{array}$$

Here $L^\infty_\lambda = \{x : \mathbb{R}_+ \to \mathbb{R} \mid ||x||^\infty_\lambda < \infty\}$ with $||x||^\infty_\lambda = \text{esssup}_{t\geq 0}|x(t)|e^{\lambda t}$.

Proposition 2 *The function Φ_α is continuous with respect to ν and C^1 with respect to h. Moreover, there exists a function Θ_α such that $\forall 0 < \lambda < \lambda_\alpha$, $\Theta_\alpha \in L^1_\lambda$ and*

$$\forall c \in L^\infty_\lambda,\ D_h\Phi(\nu^\infty_\alpha, 0) \cdot c = c - \Theta_\alpha * c.$$

The function Θ_α is known explicitly in term of f, b and α.

Let $\widehat{\Theta}_\alpha(z)$ the Laplace transform of Θ_α.
Theorem 4 *Assume all the complex roots of $z \mapsto \widehat{\Theta}_\alpha(z) - 1$ are located on the left half-plane. Then the invariant measure ν^∞_α is locally stable.*

Examples for which Theorem 4 applies. For J small enough it is always satisfied. If $b \equiv 0$, it is satisfied (and the invariant measure is always locally stable, whatever the value of the weight J). When Theorem 4 **does not** apply, it means that spontaneous oscillations may exists!

REFERENCES

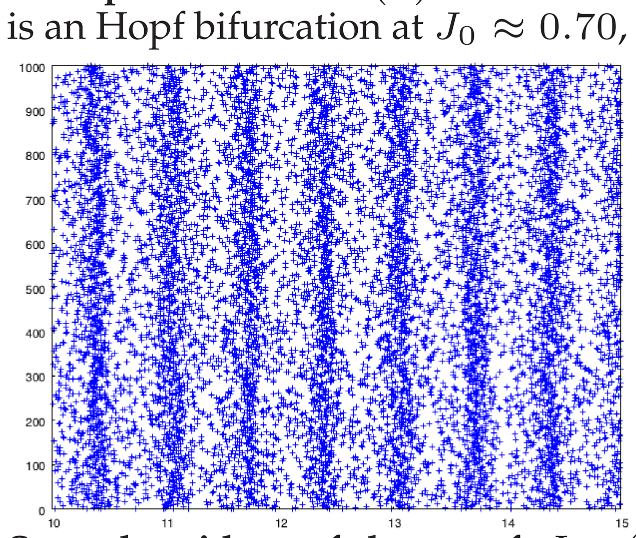
[1] De Masi, A., Galves, A., Löcherbach, E., Presutti, E., 2015. Hydrodynamic Limit for Interacting Neurons.
[2] Fournier, N., Löcherbach, E., 2016. On a toy model of interacting neurons.
[3] Cormier, Q., Tanré, E., Veltz, R., 2019. Long time behavior of a mean-field model of interacting neurons.

6. SPONTANEOUS OSCILLATIONS: HOPF BIFURCATION

A3. Assume there is some α_0 s.t. $\widehat{\Theta}_{\alpha_0}$ has two (simple) complex roots i/β_0 and $-i/\beta_0$. Assume moreover that the roots of $\widehat{\Theta}_\alpha$ crosses the imaginary axis with "non vanishing speed" at $\alpha = \alpha_0$. Let $J_0 := \alpha_0/\gamma(\alpha_0)$.

Theorem 5 (Existence of periodic solutions) *The mean-field equation* (1) *admits a family of periodic solutions in the neighborhoods of $\nu^\infty_{\alpha_0}$. The family can be parametrized by J, for J close to J_0. Their amplitudes are small (null in the limit of $J = J_0$) and their periods are close to $2\pi\beta_0$.*

Example. Consider $b(x) = 2 - 2x$ and $f(x) = x^{10}$. Then there is an Hopf bifurcation at $J_0 \approx 0.70$, for which $\beta_0 \approx 0.17$.



A raster plot (each dot corresponds to a spike of a neuron at a given time in the network). Simulation with $N = 1000$ neurons and $J = 1$. Spontaneous (stable) oscillations occurs.

Some key ideas of the proof. Let $(a_t)_{t\in\mathbb{R}}$ be a T-periodic current. We define the "asymptotic" (periodic) jump rate to be

$$\forall t \in \mathbb{R},\ \rho_a(t) := \lim_{k\in\mathbb{N},\ k\to\infty} r^\nu_a(t, -kT).$$

It solves for all t

$$\rho_a(t) = \int_{-\infty}^t K_a(t,s)\rho_a(s)ds, \quad 1 = \int_{-\infty}^t H_a(t,s)\rho_a(s)ds.$$

Probabilistic interpretation of ρ_a. Let $(Y^{\nu,a}_t)_{t\geq 0}$ be the solution of (2), driven by the T-periodic current a. Define $(\tau_i)_{i\geq 0}$ the times of its successive jumps. Let:

$$\phi_i := \tau_i - \lfloor \tfrac{\tau_i}{T} \rfloor, \quad \tau_{i+1} - \tau_i =: \Delta_i T + \phi_{i+1} - \phi_i.$$

Then, $(\phi_i)_{i\geq 1}$ is Markov with transition probability kernel $K^T_a(\cdot, s)$

$$K^T_a(t,s) = \sum_{k\geq 0} K_a(t, s - kT).$$

Let $\tilde{\phi}_a$ be the unique invariant measure of this Markov chain and let $c_a := \mathbb{E}_{\phi_i\sim\tilde{\phi}_a}\Delta_i$. Then

$$\forall t \in [0,T],\ \rho_a(t) = \frac{\tilde{\phi}_a(t)}{c_a}.$$

A difficulty. The period T itself is unknown. We define for all $\beta > 0$ and a 2π-periodic:

$$\tilde{\rho}(\beta, a) := t \mapsto \rho_d(\beta t) \quad \text{with} \quad d(t) = a(t/\beta).$$

To find periodic solutions of (1), it suffices to find roots of

$$\begin{array}{rccc} G_\alpha : C_{2\pi} \times \mathbb{R}^*_+ \times \mathbb{R}^*_+ & \to & C_{2\pi} \\ (x, \kappa, \alpha) & \mapsto & (\alpha + x) - \frac{\alpha}{\gamma(\alpha)}\tilde{\rho}(\beta, \alpha + x), \end{array}$$

Perspectives. Give an efficient algorithm to compute the stability of the invariant measures (based on Theorem 4). Study the stability of the periodic solutions. Extend the model to multi-populations (inhibitory and excitatory neurons). Study some variants of the model with, for instance, a Brownian motion in the dynamic.