A Mean-Field model of interacting neurons

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1. The model

Model neurons at the membrane potential level and study macroscopic quantities such as the activity rate of large assemblies of neurons. Study the **stability** of the network and in particular its long time behavior. **Oscillations** ? **Steady-states** ?

We consider $N \geq 1$ neurons **Integrate and Fire**. Each neuron is characterized by its membrane potential $(V_t^i)_{t\geq 0}$, $i \in 1,..,N$.

Between the spikes, $(V_t^i)_{t\geq 0}$ solves:

$$dV_t = b(V_t)dt$$

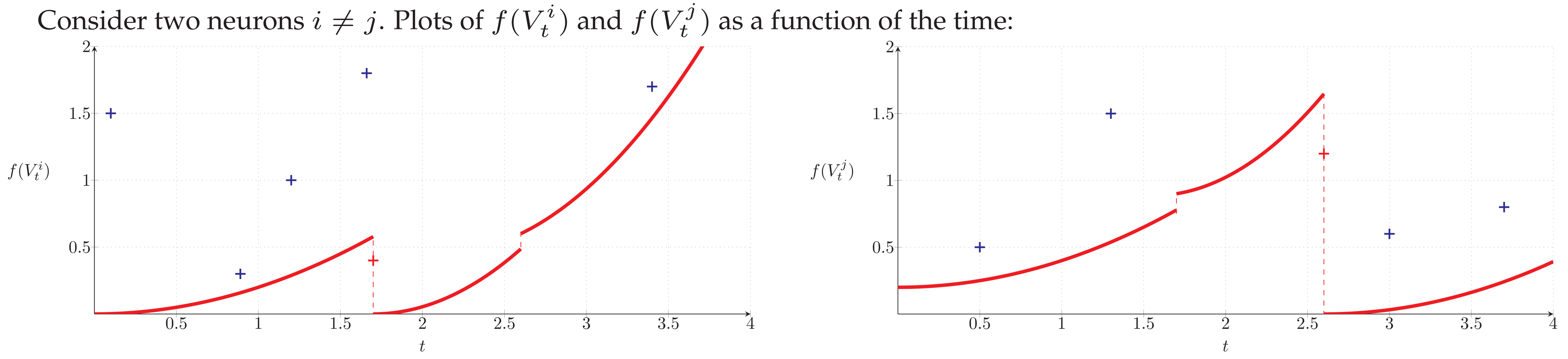
The neurons **spike randomly** with a rate $f(V_t^i)$.
When neuron i spikes at time t:

1. The potential of neuron i is reset to zero: $V_t^i = 0$
2. The other neurons $j \neq i$ receives a kick: $V_t^j = V_{t-}^j + \frac{J}{N}$

What is f ? It is a parameter of the problem. We assume that $f: \mathbb{R}_+ \to \mathbb{R}_+$ satisfied some technical assumptions fulfilled by: $f(x) = (x/\xi)^p$, $p \geq 1$.
About b: we choose $\forall x \geq 0: b(x) := \mu > 0$.

2. Example

Consider two neurons $i \neq j$. Plots of $f(V_t^i)$ and $f(V_t^j)$ as a function of the time:



3. Mean-Field limit

Assume the initial conditions $V_0^1, \ldots V_0^N$ are i.i.d. with law ν. When $N \to +\infty$, it has been proven that $\mathcal{L}((V_t^i)_{t\geq 0}) \to_N \mathcal{L}((V_t)_{t\geq 0})$ where V is the solution to the following **Mckean-Vlasov** equation:

$$V_t = V_0 + \int_0^t b(V_s)ds + J\int_0^t \mathbb{E}\, f(V_s)ds - \int_0^t \int_{\mathbb{R}_+} V_{s-}\mathbb{1}_{\{z\leq f(V_{s-})\}} N(ds,dz), \qquad (1)$$

where $N(ds,dz)$ is a Poisson measure on $\mathbb{R}_+^2$ of intensity $dsdz$ and $\mathcal{L}(V_0) = \nu$. We assume ν satisfies: $\int_0^\infty f^2(x)\nu(dx) < \infty$.

4. Existence & uniqueness

Theorem 1 *The mean-field SDE (1) has an unique (strong) solution $(V_t)_{t\geq 0}$. Moreover, it holds that:*

$$\forall t \geq 0: \mathbb{E}\, f(V_t) \leq \bar{p},$$

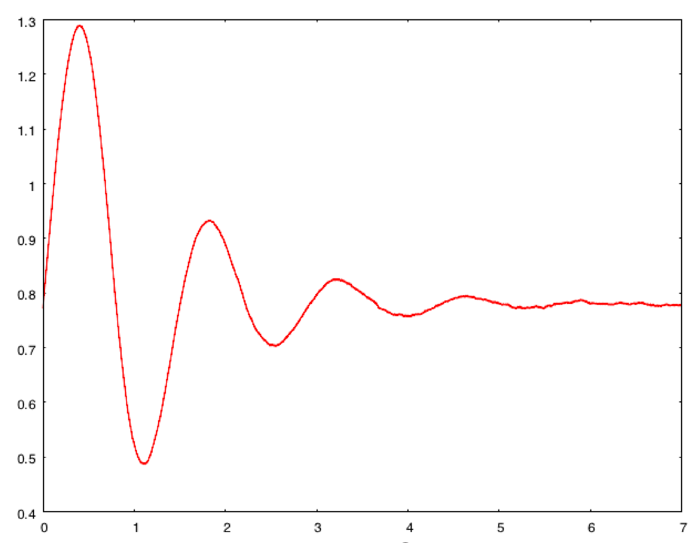
for some $0 < \bar{p} < \infty$. $\bar{p}$ only depends on f, the external current μ, the connection strength J and the initial condition ν.

5. Volterra equation

The difficulty: there is no closed formula for $t \mapsto \mathbb{E}\, f(V_t)$. The Ito formula gives:

$$\frac{d}{dt}\mathbb{E}\, f(V_t) = \mathbb{E}\, f'(V_t)(\mu + J\,\mathbb{E}\, f(V_t)) - \mathbb{E}\, f^2(V_t).$$

In particular $\frac{d}{dt}\mathbb{E}\, f(V_t)|_{t=0}$ depends on $\mathbb{E}\, f(V_0)$ but also on $\mathbb{E}\, f'(V_0)$ and $\mathbb{E}\, f^2(V_0)$.



$t \mapsto \mathbb{E}\, f(V_t)$ for $J = 0$, $f(x) = x^3$ starting from a dirac mass δ_{v_0}, $v_0 = 0.91$.

We derive a non-convolution non linear **Volterra equation** satisfied by $t \mapsto \mathbb{E}\, f(V_t)$. This Volterra equation **takes into account the initial condition** ν.

Theorem 2 *$r(t) := \mathbb{E}\, f(V_t)$ is the solution of the following N-L Volterra equation:*

$$r(t) = K^\nu(t) + \int_0^t K^0(t,u)r(u)du,$$

with:

$$K^x(t,s) := f(x + A_t - A_s)\exp\left(-\int_s^t f(x + A_u - A_s)du\right),$$

$$K^\nu(t) := \int_0^\infty K^x(t,0)\nu(dx) \text{ and } A_t := \mu t + J\int_0^t r(u)du.$$

When $J = 0$, this reduces to a **linear convolution Volterra equation**. We used **Laplace transform** to study it. The asymptotic of $t \mapsto r(t)$ is related to the location of the zeros of the Laplace transform of $t \mapsto H(t)$, with:

$$H(t) := \exp\left(-\int_0^t f(au)du\right)$$

6. Invariant measure

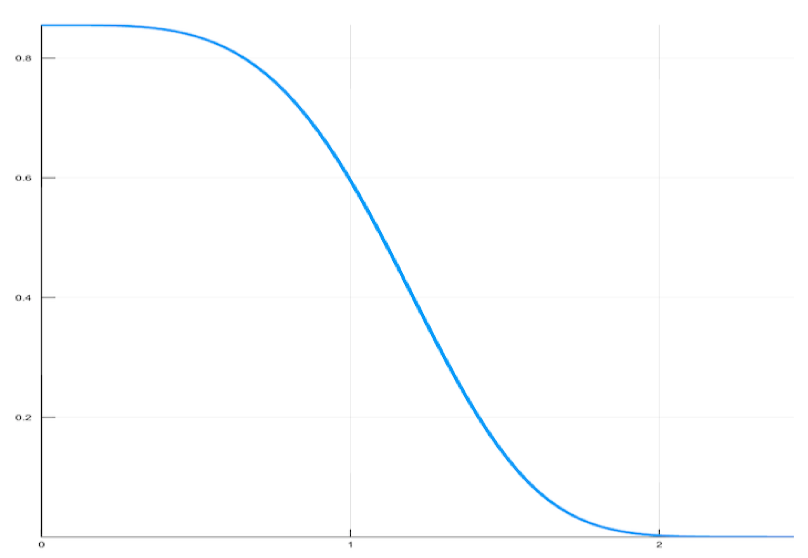
Theorem 3 *The mean-field SDE (1) admits an unique invariant measure given by:*

$$\nu_\infty(x)dx := \frac{\gamma(a^*)}{a^*}\exp\left(-\int_0^x \frac{f(y)}{a^*}dy\right)dx,$$

where:

1. *$\gamma(a^*)$ is the normalizing factor,*
2. *a^* is the unique solution of the non linear scalar equation $\frac{a^* - \mu}{\gamma(a^*)} = J$.*

Shape: $x \mapsto \nu_\infty(x)$ for $\mu = 0.1$, $J = 1.0$:



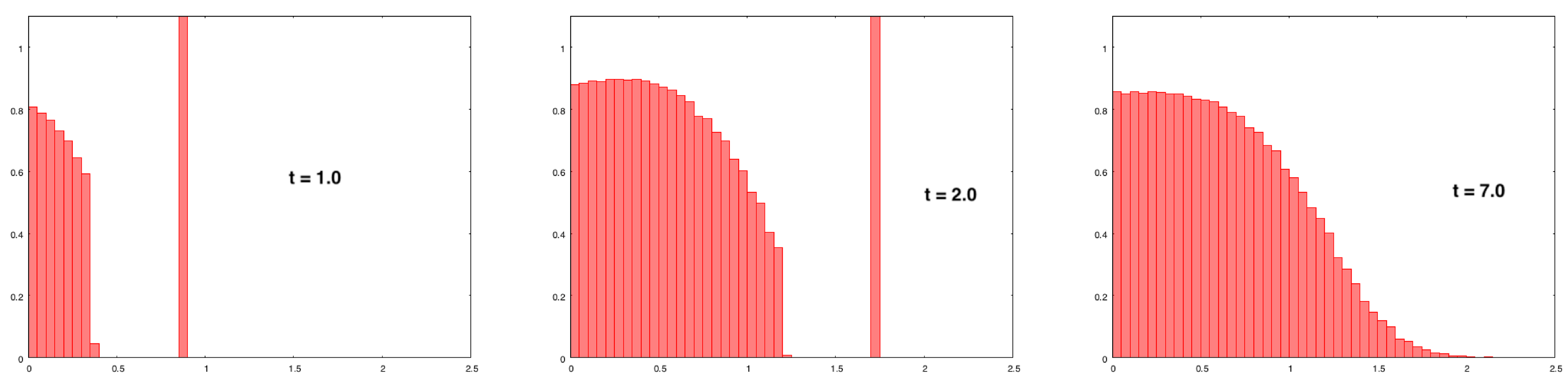
7. Convergence to the invariant measure

When J is small enough, there is a **desynchronization phenomena**:

Theorem 4 *Given $\mu > 0$, there exists $\bar{J} > 0$ such that given any initial condition ν and $0 \leq J \leq \bar{J}$, then: $\mathcal{L}(V_t) \to_{t\to\infty} \nu_\infty$. Moreover, if $\phi: \mathbb{R}_+ \to \mathbb{R}_+$ is any continuous bounded test function:*

$$|\,\mathbb{E}\,\phi(V_t) - \int_0^\infty \phi(x)\nu_\infty(dx)| \leq De^{-\lambda t},$$

for some $\lambda > 0$ only depending on f, μ and J and some explicit $D \geq 0$.



Example: initial measure $\nu = \delta_{0.5}$, $J = 1.0$, $\mu = 0.1$: evolution of the law of V_t as t goes to infinity.

8. Ergodic theorems

Theorem 5 *Assume that the conclusion of theorem 4 holds. Let $\phi: \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous bounded function. Then:*

$$\frac{1}{t}\int_0^t \phi(V_s)ds \xrightarrow[t\to\infty]{a.s.} \int_0^\infty \phi(x)\nu_\infty(dx).$$

Moreover:

$$\sqrt{t}\left[\frac{1}{t}\int_0^t \phi(V_s)ds - \int_0^\infty \phi(x)\nu_\infty(dx)\right] \xrightarrow[t\to\infty]{\mathcal{L}} \mathcal{N}(0,\sigma^2),$$

where $\sigma^2 := \int_0^\infty [u(0) - u(y)]^2 f(y)\nu_\infty(dy)$ and u is the unique solution of the following Poisson ODE:

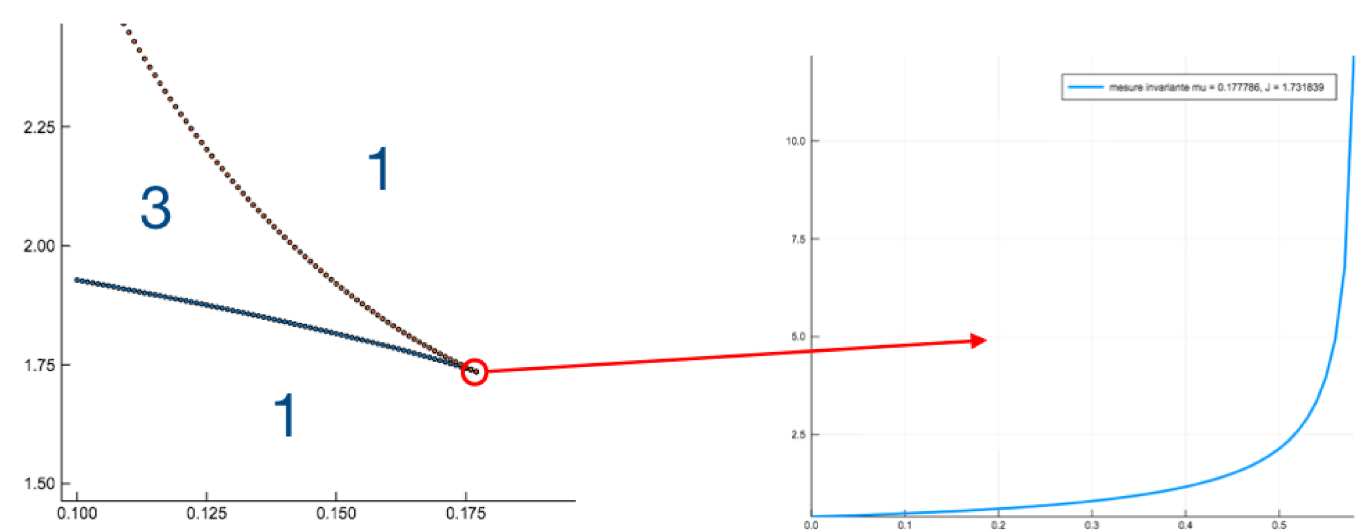
$$\mathcal{L}u = \phi - \int_0^\infty \phi d\nu_\infty, \quad \int_0^\infty u d\nu_\infty = 0,$$

and $\mathcal{L}u = a^* u'(x) + [u(0) - u]f$.

Application: an inverse problem to **estimate** J and μ using one single electrode plugged into one neuron of the network.

9. Open questions:

- What happens for $J > \bar{J}$? Is the invariant measure locally stable?
- Assume now $b(x) = \mu - x$. In that case, depending on $\mu > 0$ and J, there can be up to **three non-trivial invariant measures. Which of them are stable? Oscillations in that case?**



Cusp bifurcation at the point $\mu \approx 0.17$, $J \approx 1.73$.

- **Excitatory and inhibitory neurons**: a coupled system of two mean-field equations. Invariant measures? Oscillations? **Numerically, yes!**

References

[1] De Masi, A., Galves, A., Löcherbach, E., Presutti, E., 2015. Hydrodynamic Limit for Interacting Neurons.

[2] Fournier, N., Löcherbach, E., 2016. On a toy model of interacting neurons.

[3] Drogoul, A., Veltz, R., 2017. Hopf bifurcation in a nonlocal nonlinear transport equation stemming from stochastic neural dynamics