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A mean-field model of Integrate-and-Fire neurons: non-linear stability of the stationary solutions

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Abstract

We investigate a stochastic network composed of Integrate-and-Fire spiking neurons, focusing on its mean-field asymptotics. We consider an invariant probability measure of the McKean-Vlasov equation and establish an explicit sufficient condition to ensure the local stability of this invariant distribution. Furthermore, we prove a conjecture proposed initially by J. Touboul and P. Robert regarding the bistable nature of a specific instance of this neuronal model.

Keywords: McKean-Vlasov SDE; Long-time behavior; Mean-field interaction; Volterra integral equation; Piecewise deterministic Markov process .
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1 Introduction

We consider a network of N spiking neurons. Each neuron is characterized by its membrane potential $(X_t^{i,N})_{t\geq 0}$. Each neuron emits "spikes" randomly, at a rate $f(X_t^{i,N})$, which only depends on its membrane potential. The function $f : \mathbb{R} \to \mathbb{R}_+$ is deterministic. When a neuron spikes (say neuron i spikes at time τ), its potential is instantaneously reset to zero (we say zero is the *resting value*) while the other neurons receive a small kick:

$$X_{\tau_+}^{i,N} = 0, \quad \text{and} \quad \forall j \neq i, \quad X_{\tau_+}^{j,N} = X_{\tau_-}^{j,N} + J_{i\to j}^N.$$

In this equation, the *synaptic weight* $J_{i\to j}^N$ is a deterministic constant that models the interaction between the neurons i and j. Finally, between the spikes, each neuron follows its own dynamics given by the scalar ODE:

$$\dot{X}_t^{i,N} = b(X_t^{i,N}),$$

where $b : \mathbb{R} \to \mathbb{R}$ is a deterministic function. We say that b models the *sub-threshold* dynamics of the neuron. We are interested in the dynamics of one particle (say $(X_t^{1,N})$) in the limit where the number of particles N goes to infinity. To simplify, we assume that the neurons are all-to-all connected with the same weight:

$$\forall i, j,\ i \neq j \quad J_{i\to j}^N = \frac{J}{N}.$$

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In this work, the deterministic constant J is non-negative (when f is non-decreasing, we say it is an *excitatory network*). At the initial time, all the neurons start with *i.i.d.* initial conditions with law $\nu \in \mathcal{P}(\mathbb{R}_+)$. We assume that $b(0) \geq 0$, so the membrane potentials stay on $\mathbb{R}_+$.

One expects *propagation of chaos* to hold: as N goes to infinity, any pair of neurons of the network (say $X_t^{1,N}$ and $X_t^{2,N}$) become more and more independent, and each neuron (say $(X_t^{1,N})$) converges in law to the solution of the following McKean-Vlasov equation:

$$X_t^\nu = X_0^\nu + \int_0^t b(X_s^\nu)\mathrm{d}s + J\int_0^t \mathbb{E}f(X_s^\nu)\mathrm{d}s - \int_0^t \int_{\mathbb{R}_+} X_{s-}^\nu \mathbb{1}_{\{z \leq f(X_{s-}^\nu)\}} N(\mathrm{d}s, \mathrm{d}z). \tag{1.1}$$

In this equation, $N(\mathrm{d}s, \mathrm{d}z)$ is a Poisson measure on $\mathbb{R}_+^2$ with intensity the Lebesgue measure $\mathrm{d}s\mathrm{d}z$. In addition, the initial condition X_0^ν has law $\nu \in \mathcal{P}(\mathbb{R}_+)$ and is independent of the Poisson measure. Informally, eq. (1.1) can be understood in the following way: between the jumps, (X_t^ν) solves the ODE $\dot{X}_t^\nu = b(X_t^\nu) + J\mathbb{E}f(X_t^\nu)$ and (X_t^ν) jumps to zero at a rate $f(X_t^\nu)$.

This model of neurons is sometimes known in the literature as the “Escape noise”, “Noisy output”, “Hazard rate” model. We refer to [14, Ch. 9] for a review. From a mathematical point of view, it was first introduced by [7], where it is described as a time-continuous version of the Galves–Löcherbach model [13]. Under few assumptions on b, f, and on the initial condition ν, it is known that (1.1) is well-posed: see [7] (with the assumption that the initial condition ν is compactly supported), [12] (assuming only that ν has a first moment) and [4] (where a different proof is given, based on the renewal structure of the equation, see Theorem 2.2 below). The convergence of the finite particle system $(X_t^{i,N})$ to the solution of (1.1) is studied in [12] where the rate of convergence, of the order $C(t)/\sqrt{N}$, is also given.

When the number of neurons is finite, the network $(X_t^{i,N})$ is a Markov process (it is a Piecewise Deterministic Markov Process, see [6]). So under quite general assumptions on b, and f, this $\mathbb{R}_+^N$-valued process has a unique invariant probability measure, which is globally attractive. We refer to [10], [20], [16] and [17] for studies about the long time behavior of the finite particle system.

The long time behavior of the solution of the limit equation (1.1) is more complex, essentially because this is a McKean-Vlasov equation, and so it is not Markovian. In particular, (1.1) may have multiple invariant probability measures (see [20, 4] and Section 4 below for explicit examples). Even when the invariant probability measure is unique, it is not necessarily attractive. In [8, 5], the authors show that a Hopf bifurcation might appear when the interaction parameter J varies, leading to periodic solutions of (1.1). The specific case $b \equiv 0$ is studied in [12]. It is proved that for all $J > 0$, there are precisely two invariant probability measures: the Dirac δ_0, which is unstable, and a non-trivial one, which is globally attractive. This case $b \equiv 0$ is also investigated in [9], where the authors prove that the non-trivial invariant measure is locally attractive with an exponential rate of convergence. Both [12] and [9] rely on the Fokker-Planck PDE satisfied by the density of the solution of (1.1). Finally, in [4], general conditions are given on b and f such that the McKean-Vlasov equation (1.1) admits a globally attractive invariant measure, assuming that the interaction parameter J is small enough. For such weak enough interactions, similar results have been obtained for variants of this model, such as the time-elapsed model (see [19]) or the Integrate-and-fire with a fixed deterministic threshold (see [1], [2] and [11] for another “Poissonian” variant). Finally, in [18], the authors consider the case $b(x) = -x$, $f(x) = k\min(x, \lambda)$ for some constants $k, \lambda > 0$. They obtain a bistability result using a coupling method and study the metastable behavior of the particle system.

Understanding the long time behavior of (1.1) for an arbitrary interaction parameter J is a challenging open question. In this work, we address the question of the local stability of an invariant probability measure of (1.1), without assumptions on the size of the interactions J. We say that $\nu_\infty \in \mathcal{P}(\mathbb{R}_+)$ is an invariant probability measure of (1.1) if for all $t \geq 0$, it holds that the law of $X_t^{\nu_\infty}$, solution of (1.1), is equal to ν_∞. We are interested in the stability of such an invariant probability measure ν_∞. Our main contribution is to provide an explicit criterion to decide stability. If this criterion is satisfied, then for any $\nu \in \mathcal{P}(\mathbb{R}_+)$ sufficiently closes to ν_∞, the law of X_t^ν converges to ν_∞, at an exponential rate.

Recently, in [3], this question has been addressed for McKean-Vlasov with smooth coefficients driven by Brownian motions. It is shown that the stability is governed by the location of the zeros of a particular analytic function associated to the dynamics. We follow and adapt to (1.1) the strategy introduced in [3]. One key difficulty is to identify a distance on probability measures that is well-suited to the particular structure of (1.1). While the Wasserstein W_1 distance is used in [3] for McKean-Vlasov driven by Brownian motions, we use here instead the "Bounded Lipschitz" distance, see eq. (1.2) below. Using this distance, we first prove a stability result of (1.1) on a finite time interval $[0, T]$ (Theorem 2.2). Using the same distance, we then study the long time behavior of an auxiliary Markov process, see Proposition 2.5. Combining these two results, we derive our main result, Theorem 2.8, which provides an explicit criterion to decide if an invariant probability measure is stable and quantifies the convergence. We then identify in Proposition 2.9 a structural condition on the coefficients, namely $f + b' \geq 0$, such that eq. (1.1) has a unique, locally stable, invariant probability measure. This generalizes the results of [12] and [9], valid for $b \equiv 0$.

Finally, we study an explicit example where bistability occurs. We study the case:

$$f(x) = x^2 \quad \text{and} \quad b(x) = -x, \quad \forall x \in \mathbb{R}_+.$$

This example is studied numerically in [20]. In this work, the authors conjecture that (1.1) exhibits a phase transition: there is a parameter $J_* > 0$ such that for all $J < J_*$, (1.1) has a unique invariant probability measure (the Dirac mass at zero) while for all $J > J_*$, (1.1) has three invariant probability measures (the Dirac mass at zero and two other non-trivial probability measures). We provide a proof of this conjecture; see Proposition 4.4. Provided that $J > J_*$, an exact analysis of the stability of the two non-trivial stationary distributions seems impossible. However, a local analysis near the bifurcation points $J = J_*$ is possible. Then, in view of Theorem 2.8, we conjecture that one is stable and the other is unstable. As the Dirac probability measure is stable (see [20]), we deduce that bistability is the main paradigm for $J > J_*$.

Main notations. We write $\mathcal{P}(\mathbb{R}_+)$ for the space of probability measures on $\mathbb{R}_+$. Given $\nu \in \mathcal{P}(\mathbb{R}_+)$ and $g : \mathbb{R}_+ \to \mathbb{R}$ a test function, we write $\langle g, \nu \rangle = \int_{\mathbb{R}_+} g(x)\nu(\mathrm{d}x)$. For Z a random variable on $\mathbb{R}_+$, we write $Law(Z) \in \mathcal{P}(\mathbb{R}_+)$ for its probability law. We denote by $Lip_1(\mathbb{R}_+)$ the space of globally Lipschitz functions from $\mathbb{R}_+$ to $\mathbb{R}$ with Lipschitz norm bounded by 1. We equip $\mathcal{P}(\mathbb{R}_+)$ with the following distance: for all $\nu, \mu \in \mathcal{P}(\mathbb{R}_+)$,

$$||\nu - \mu||_0 := \sup \left\{ \int_{\mathbb{R}_+} g\mathrm{d}(\nu - \mu); \quad g \in Lip_1(\mathbb{R}_+), \quad \sup_{x \in \mathbb{R}_+} |g(x)| \leq 1 \right\}. \tag{1.2}$$

2 Main results

Let $N(\mathrm{d}s, \mathrm{d}z)$ be a Poisson measure on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity the Lebesgue measure $\mathrm{d}s\mathrm{d}z$. Given $f : \mathbb{R}_+ \to \mathbb{R}_+$, $b : \mathbb{R}_+ \to \mathbb{R}$ and $J \geq 0$, we consider the McKean-Vlasov SDE (1.1), where at the initial time, X_0^ν is distributed according to law $\nu \in \mathcal{P}(\mathbb{R}_+)$.

Assumption 2.1. *We assume that $b, f \in C^1(\mathbb{R}_+;\mathbb{R}_+)$ with $\|f\|_\infty + \|f'\|_\infty < \infty$ and that b is globally Lipschitz, with $b(0) \geq 0$.*

The condition $b(0) \geq 0$ ensures that the dynamics stays on $\mathbb{R}_+$. We first show the following existence and stability result on a finite horizon time $[0,T]$:

Theorem 2.2. *Under Assumption 2.1, the mean-field equation (1.1) has a unique path-wise solution for all $\nu \in \mathcal{P}(\mathbb{R}_+)$. In addition, for all $T > 0$, there exists a constant C_T such that for all $t \in [0,T]$:*

$$\forall \nu, \mu \in \mathcal{P}(\mathbb{R}_+), \quad \|Law(X_t^\nu) - Law(X_t^\mu)\|_0 \leq C_T \|\nu - \mu\|_0.$$

Our second result concerns the ergodic behavior of the following associated linear equation. Let $\alpha > 0$. Denote by $(Y_t^{\alpha,\nu})$ the solution of

$$Y_t^{\alpha,\nu} = Y_0^{\alpha,\nu} + \int_0^t b(Y_s^{\alpha,\nu})\mathrm{d}s + \alpha t - \int_0^t \int_{\mathbb{R}_+} Y_{s-}^{\alpha,\nu} \mathbb{1}_{\{z \leq f(Y_{s-}^{\alpha,\nu})\}} N(\mathrm{d}s, \mathrm{d}z), \tag{2.1}$$

with $Law(Y_0^{\alpha,\nu}) = \nu$. That is, we have "frozen" the non-linear interactions $J\mathbb{E}f(X_t^\nu)$ of (1.1), and we have replaced it by the constant drift α.

We denote by $\varphi_t^\alpha(x)$ the unique solution of the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t^\alpha(x) = b(\varphi_t^\alpha(x)) + \alpha, \quad \text{with} \quad \varphi_0^\alpha(x) = x.$$

In what follows, we assume that

Assumption 2.3. *The triple (b, f, α) satisfies*

1. *The jump rate is asymptotically lower bounded:*

$$\underline{\lambda} := \liminf_{t\to\infty} \inf_{x \geq 0} \frac{1}{t} \int_0^t f(\varphi_s^\alpha(x))\mathrm{d}s > 0. \tag{2.2}$$

2. *There exists a constant C such that for all $t \geq 0$ and $x, y \in \mathbb{R}_+$:*

$$|\varphi_t^\alpha(x) - \varphi_t^\alpha(y)| \leq C|x - y|.$$

Remark 2.4. The constants $\underline{\lambda}$ and C are allowed to depend on α. The first point is satisfied if $f(x) \geq f_{\min} > 0$. The second point is satisfied if $b(x) = b_0 - b_1 x$ where $b_0, b_1 \geq 0$. Indeed, in that case, the flow is:

$$\varphi_t^\alpha(x) = \begin{cases} \left(x - \frac{b_0+\alpha}{b_1}\right) e^{-b_1 t} + \frac{b_0+\alpha}{b_1} & \text{if } b_1 > 0, \\ x + (b_0 + \alpha)t & \text{if } b_1 = 0. \end{cases} \tag{2.3}$$

We have:

Proposition 2.5. *Under Assumption 2.1 and 2.3, there exists $C_* > 0$ and $\lambda_* \in (0, \underline{\lambda})$ such that for all $\nu, \mu \in \mathcal{P}(\mathbb{R}_+)$, and for all $t \geq 0$*

$$\|Law(Y_t^{\alpha,\nu}) - Law(Y_t^{\alpha,\mu})\|_0 \leq C_* e^{-\lambda_* t} \|\nu - \mu\|_0.$$

Therefore, (2.1) has a unique invariant probability measure. As shown in [4], this invariant probability measure is:

$$\nu_\infty^\alpha(x)\mathrm{d}x = \frac{\gamma(\alpha)}{b(x) + \alpha} \exp\left(-\int_0^x \frac{f(y)}{b(y) + \alpha}\mathrm{d}y\right) \mathbb{1}_{[0,\sigma_\alpha)}(x)\mathrm{d}x, \tag{2.4}$$

where the support $\sigma_\alpha \in (0, \infty]$ is given by

$$\sigma_\alpha = \inf\{y \geq 0; \quad b(y) + \alpha = 0\} = \lim_{t \to \infty} \varphi^\alpha_t(0).$$

The changes of variables $y = \varphi^\alpha_u(0)$ and $x = \varphi^\alpha_t(0)$ in (2.4) show that for any bounded measurable test function g,

$$\langle g, \nu^\alpha_\infty \rangle = \gamma(\alpha) \int_0^\infty g(\varphi^\alpha_t(0)) \exp\left(- \int_0^t f(\varphi^\alpha_u(0)) \mathrm{d}u \right) \mathrm{d}t.$$

In particular, choosing $g = f$, we find that $\gamma(\alpha)$ satisfies:

$$\gamma(\alpha) = \int_{\mathbb{R}_+} f(x) \nu^\alpha_\infty(x) \mathrm{d}x.$$

We say that ν_∞, an invariant probability measure of (1.1), is non-trivial if $J\langle f, \nu_\infty \rangle \neq 0$. Note that there is a one-to-one correspondence between the non-trivial invariant probability measures of (1.1) and the $\alpha > 0$ satisfying:

$$\alpha = J\gamma(\alpha).$$

Let ν_∞ be a non-trivial invariant probability measure of (1.1). Let $\alpha = J\langle f, \nu_\infty \rangle$ such that $\nu_\infty = \nu^\alpha_\infty$. We assume that the triple (b, f, α) satisfies Assumption 2.3. To state our main result on the stability of ν_∞, we need to introduce two notations. We define for all $t \geq 0$

$$H_\alpha(t) := \exp\left(- \int_0^t f(\varphi^\alpha_s(0)) \mathrm{d}s \right), \tag{2.5}$$

$$\Psi_\alpha(t) := \alpha \int_0^\infty H_\alpha(t+u) \frac{f(\varphi^\alpha_{t+u}(0)) - f(\varphi^\alpha_u(0))}{b(\varphi^\alpha_u(0)) + \alpha} \mathrm{d}u. \tag{2.6}$$

Let $\lambda_* > 0$ be the constant given by Proposition 2.5.

Assumption 2.6. *Assume that* $f \in C^2(\mathbb{R}_+; \mathbb{R}_+)$ *with* $\|f^{(k)}\|_\infty < \infty$ *for* $k \in \{0, 1, 2\}$. *Assume* $b \in C^2(\mathbb{R}_+; \mathbb{R})$ *with* $b(0) \geq 0$ *and* $\|b'\|_\infty + \|b''\|_\infty < \infty$. *Finally, assume there exists a constant* $\lambda_\alpha \in (0, \lambda_*)$ *such that*

$$\sup_{t \geq 0} e^{\lambda_\alpha t} \int_0^\infty H_\alpha(t+u) \left| \frac{f(\varphi^\alpha_{t+u}(0)) - f(\varphi^\alpha_u(0))}{b(\varphi^\alpha_u(0)) + \alpha} \right| \mathrm{d}u < \infty.$$

Remark 2.7. This last estimate holds if $b(x) = b_0 - b_1 x$ with $b_1 \geq 0$. This follows from the explicit expression of $\varphi^\alpha_t(0)$, see (2.3), and from the fact that f is globally Lipschitz.

We write $D_\alpha := \{z \in \mathbb{C},\ \Re(z) > -\lambda_\alpha\}$. We consider for $z \in D_\alpha$

$$\hat{H}_\alpha(z) := \int_0^\infty e^{-zt} H_\alpha(t) \mathrm{d}t,$$

the Laplace transform of H_α. Similarly, let $\hat{\Psi}_\alpha(z)$ be the Laplace transform of Ψ_α. By assumption, the two functions are analytic on D_α. Our main result is

Theorem 2.8. *Let* ν_∞ *be a non-trivial invariant probability measure of* (1.1). *Let* $\alpha = J\langle f, \nu_\infty \rangle \in \mathbb{R}^*_+$. *In addition to Assumptions 2.3 and 2.6, assume that*

$$-\lambda'_\alpha := \sup\{\Re(z); \quad z \in D_\alpha, \quad \hat{H}_\alpha(z) = \hat{\Psi}_\alpha(z)\} < 0. \tag{2.7}$$

Then there exists $C, \epsilon > 0$ *and* $\lambda \in (0, \lambda'_\alpha)$ *such that for all initial condition* $\nu \in \mathcal{P}(\mathbb{R}_+)$ *with* $\|\nu - \nu_\infty\|_0 < \epsilon$, *it holds that*

$$\forall t \geq 0, \quad \|Law(X^\nu_t) - \nu_\infty\|_0 \leq Ce^{-\lambda t} \|\nu - \nu_\infty\|_0.$$

We provide in Section 4 an explicit example with multiple invariant probability measures. The spectral assumption $\lambda'_\alpha > 0$ is automatically satisfied if the structural condition $f + b' \geq 0$ holds. More precisely,

Proposition 2.9. *We have:*

1. *Let ν_∞ be a non-trivial invariant probability measure of (1.1). Let $\alpha = J\langle f, \nu_\infty\rangle \in \mathbb{R}_+^*$. In addition to Assumption 2.3 and 2.6, assume that $f + b' \geq 0$ on $[0, \sigma_\alpha)$. Then $\lambda'_\alpha > 0$, and so ν_∞ is locally stable.*
2. *Grant Assumption 2.1. Assume moreover that $b \in C^1(\mathbb{R}_+)$, $b(0) > 0$ and that $f + b' \geq 0$ on $\mathbb{R}_+$. Then (1.1) has exactly one non-trivial invariant probability measure.*

3 Proofs

3.1 Notations

Let $T > 0$. Given $a \in C([0,T];\mathbb{R}_+)$, we denote by $Y^{a,\nu}_{t,s}$ the solution of the linear non-homogeneous SDE

$$Y^{a,\nu}_{t,s} = Y^{a,\nu}_{s,s} + \int_s^t b(Y^{a,\nu}_{u,s})\mathrm{d}u + \int_s^t a_u \mathrm{d}u - \int_s^t \int_{\mathbb{R}_+} Y^{a,\nu}_{u-,s} \mathbb{1}_{\{z \leq f(Y^{a,\nu}_{u-,s})\}} N(\mathrm{d}u, \mathrm{d}z), \tag{3.1}$$

where at time s, $Law(Y^{a,\nu}_{s,s}) = \nu$. We let $\varphi^a_{t,s}(x)$ be the solution of the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi^a_{t,s}(x) = b(\varphi^a_{t,s}(x)) + a_t, \quad \varphi^a_{s,s}(x) = x.$$

As in [4], we denote by $K^\nu_a(t,s)$ the density of the first jump of $(Y^{a,\nu}_{t,s})_{t\geq s}$:

$$K^\nu_a(t,s) := \int_{\mathbb{R}+} f(\varphi^a_{t,s}(x)) \exp\left(-\int_s^t f(\varphi^a_{u,s}(x))\mathrm{d}u\right)\nu(\mathrm{d}x).$$

Similarly, let $H^\nu_a(t,s)$ be the survival function of the first jump:

$$H^\nu_a(t,s) := \int_{\mathbb{R}+} \exp\left(-\int_s^t f(\varphi^a_{u,s}(x))\mathrm{d}u\right)\nu(\mathrm{d}x).$$

When a does not depend on t, that is $a \equiv \alpha \in \mathbb{R}_+$, we write for all $t, x \geq 0$:

$$H^x_\alpha(t) = H^{\delta_x}_\alpha(t,0), \quad K^x_\alpha(t) = K^{\delta_x}_\alpha(t,0), \quad \varphi^\alpha_t(x) = \varphi^\alpha_{t,0}(x). \tag{3.2}$$

3.2 Proof of Theorem 2.2

Lemma 3.1. *There exists a constant C_T such that for all $g \in C^1(\mathbb{R}_+)$, for all $a, \tilde{a} \in C([0,T];\mathbb{R}_+)$, for all $0 \leq s \leq t \leq T$,*

$$\left|\int_{\mathbb{R}_+} g(\varphi^a_{t,s}(x))H^{\delta_x}_a(t,s)\nu(\mathrm{d}x) - \int_{\mathbb{R}_+} g(\varphi^{\tilde{a}}_{t,s}(x))H^{\delta_x}_{\tilde{a}}(t,s)\mu(\mathrm{d}x)\right| \\ \leq C_T(\|g\|_\infty + \|g'\|_\infty)\left[\int_s^t |a_u - \tilde{a}_u|\mathrm{d}u + \|\nu - \mu\|_0\right].$$

Proof. Write $F_a(x) = g(\varphi^a_{t,s}(x))H^{\delta_x}_a(t,s)$. Using the explicit formula satisfied by the survival function $H^{\delta_x}_a(t,s)$, we find that F_a is $C^1(\mathbb{R}_+)$ with $\|F_a\|_\infty \leq \|g\|_\infty$ and

$$\|F'_a\|_\infty \leq [\|g'\|_\infty + \|g\|_\infty\|f'\|_\infty]\, e^{T\|b'\|_\infty}.$$

Therefore, for $C_T = (1+e^{T\|b'\|_\infty})(1+\|f'\|_\infty)$, we have:

$$|\langle F_a, \nu - \mu\rangle| \leq C_T(\|g\|_\infty + \|g'\|_\infty)\|\nu - \mu\|_0.$$

In addition, using Gronwall's inequality, we find that

$$|\varphi^a_{t,s}(x) - \varphi^{\tilde{a}}_{t,s}(x)| \leq e^{T\|b'\|_\infty} \int_s^t |a_u - \tilde{a}_u| \mathrm{d}u.$$

Using the explicit formula of F_a, we deduce that there exists another constant C_T such that

$$|F_a(x) - F_{\tilde{a}}(x)| \leq C_T(\|g\|_\infty + \|g'\|_\infty) \int_s^t |a_u - \tilde{a}_u| \mathrm{d}u.$$

Altogether, we deduce the result. □

Let $r^\nu_a(t,s) := \mathbb{E} f(Y^{a,\nu}_{t,s})$. It holds that the jump rate $r^\nu_a(t,s)$ and $K^\nu_a(t,s)$ are linked by the following Volterra integral equation [4]:

Lemma 3.2. *It holds that:*

$$r^\nu_a(t,s) = K^\nu_a(t,s) + \int_s^t r^{\delta_0}_a(t,u) K^\nu_a(u,s) \mathrm{d}u.$$

Proof. Let $t \geq s$ and $\tau_s := \inf\{u > s, Y^{a,\nu}_{u-,s} \neq Y^{a,\nu}_{u,s}\}$ be the time of the first jump of $Y^{a,\nu}_{\cdot,s}$ after s. The law of τ_s is $K^\nu_a(u,s)\mathrm{d}u$. We have

$$r^\nu_a(t,s) = \mathbb{E} f(Y^{a,\nu}_{t,s}) = \mathbb{E} f(Y^{a,\nu}_{t,s}) \mathbb{1}_{\{\tau_s \geq t\}} + \mathbb{E} f(Y^{a,\nu}_{t,s}) \mathbb{1}_{\{\tau_s \in (s,t)\}}.$$

For a fixed initial condition x, it holds that $Y^{a,\delta_x}_{t,s} = \varphi^a_{t,s}(x)$ under the event $\{\tau_s > t\}$. Therefore, the first term is equal to

$$\int_{\mathbb{R}_+} f(\varphi^a_{t,s}(x)) H^{\delta_x}_a(t,s) \nu(\mathrm{d}x) = K^\nu_a(t,s).$$

Using the strong Markov property at time τ_s and using that the process is reset to 0 after this jump, we find that the second term is equal to

$$\mathbb{E} f(Y^{a,\nu}_{t,s}) \mathbb{1}_{\{\tau_s \in (s,t)\}} = \mathbb{E} f(Y^{a,\delta_0}_{t,\tau_s} \mathbb{1}_{\{\tau_s \in (s,t)\}}) = \int_s^t r^{\delta_0}_a(t,u) K^\nu_a(u,s) \mathrm{d}u.$$

Altogether, we deduce the result. □

More generally, by the exact same argument, we have

Lemma 3.3. *Let $g : \mathbb{R}_+ \to \mathbb{R}$ be a bounded test function. It holds that for all $t \geq s$,*

$$\mathbb{E} g(Y^{a,\nu}_{t,s}) = \int_{\mathbb{R}_+} g(\varphi^a_{t,s}(x)) H^{\delta_x}_a(t,s) \nu(\mathrm{d}x) + \int_0^t K^\nu_a(u,s) \mathbb{E} g(Y^{a,\delta_0}_{t,u}) \mathrm{d}u.$$

Exploiting the Volterra integral equation of Lemma 3.2, we deduce that:

Lemma 3.4. *There exists a constant C_T such that for all $a, \tilde{a} \in C([0,T]; \mathbb{R}_+)$, it holds that*

$$\forall s \leq t \leq T, \quad |r^\nu_a - r^\mu_{\tilde{a}}|(t,s) \leq C_T \int_s^t |a_u - \tilde{a}_u| \mathrm{d}u + C_T \|\nu - \mu\|_0.$$

Proof. We first prove the inequality when $\nu = \mu = \delta_0$. To simplify the notations, we also denote by $r_a(t,s) := r_a^{\delta_0}(t,s)$ and $K_a(t,s) := K_a^{\delta_0}(t,s)$. Let $\Delta(t,s) := r_a(t,s) - r_{\tilde{a}}(t,s)$. Using Lemma 3.2, we have

$$\Delta(t,s) = (K_a - K_{\tilde{a}})(t,s) + \int_s^t (K_a - K_{\tilde{a}})(u,s) r_a(t,u) \mathrm{d}u + \int_s^t K_{\tilde{a}}(u,s)\Delta(t,u)\mathrm{d}u.$$

As $|r_a(t,s)| \leq \|f\|_\infty$ and $|K_a(t,s)| \leq \|f\|_\infty$, this shows that $(t,s) \mapsto \Delta(t,s)$ is continuous. In addition we have:

$$|\Delta(t,s)| \leq C_T \int_s^t |a_u - \tilde{a}_u| \mathrm{d}u + C_T \int_s^t |\Delta(t,u)|\mathrm{d}u.$$

We conclude by using the Grönwall's inequality. The extension to arbitrary $\nu, \mu \in \mathcal{P}(\mathbb{R}_+)$ follows from Lemma 3.1 (with $g = f$) and Lemma 3.3. ☐

By similar arguments, we have, using Lemma 3.1 and Lemma 3.3:

Lemma 3.5. *There exists a constant C_T such that for all $a, \tilde{a} \in C([0,T];\mathbb{R}_+)$, for all $g \in C^1(\mathbb{R}_+)$:*

$$\forall s \leq t \leq T, \quad |\mathbb{E}g(Y_{t,s}^{a,\nu}) - \mathbb{E}g(Y_{t,s}^{\tilde{a},\mu})| \leq C_T(\|g\|_\infty + \|g'\|_\infty)\left[\int_s^t |a_u - \tilde{a}_u|\mathrm{d}u + \|\nu - \mu\|_0\right].$$

We now give the proof of Theorem 2.2. Existence is proven exactly as in [4]. We prove the stated stability estimate, which implies uniqueness. Let (X_t^ν) and (X_t^μ) be two solutions of (1.1) starting from laws ν and μ. Let $a_t = J\mathbb{E}f(X_t^\nu)$ and $\tilde{a}_t = J\mathbb{E}f(X_t^\mu)$. Then, $a \in C([0,T];\mathbb{R}_+)$. Indeed, by Ito's formula, we have

$$a_t = J\mathbb{E}f(X_t^\nu) = J\mathbb{E}f(X_0^\nu) + \int_0^t J\mathbb{E}f'(X_s^\nu)(b(X_s^\nu) + a_s)\mathrm{d}s + \int_0^t J\mathbb{E}(f(0) - f(X_s^\nu))f(X_s^\nu)\mathrm{d}s.$$

In addition, (X_t^ν) is a solution of (3.1) with a. The same holds for (X_t^μ) with $\tilde{a}$. Therefore, by Lemma 3.4 (with $s = 0$), we deduce that

$$|a_t - \tilde{a}_t| \leq C_T \int_0^t |a_u - \tilde{a}_u|\mathrm{d}u + C_T\|\nu - \mu\|_0.$$

By Grönwall's inequality, we deduce that

$$\sup_{t\in[0,T]} |a_t - \tilde{a}_t| \leq C_T e^{C_T}\|\nu - \mu\|_0.$$

The stability estimate of Theorem 2.2 then follows from Lemma 3.5. ☐

3.3 Proof of Proposition 2.5

Recall that the Markov process $(Y_t^{\alpha,\nu})$ is defined by (2.1). We also use the notation $\mathbb{E}_x g(Y_t^\alpha) := \mathbb{E}g(Y_t^{\alpha,\delta_x})$, for all $x \geq 0$. The first step is to prove that $r_\alpha(t) := \mathbb{E}_0 f(Y_t^\alpha)$ converges to $\gamma(\alpha)$ at an exponential speed:

Lemma 3.6. *Under assumptions 2.1 and 2.3, there is a constant $\theta_\alpha > 0$ such that*

$$\sup_{t\geq 0}|r_\alpha(t) - \gamma(\alpha)|e^{\theta_\alpha t} < \infty.$$

Proof. By Lemma 3.2, r_α is the solution of a Volterra convolution equation. Therefore, the strategy is to use the Laplace transform to deduce the asymptotic behavior of $t \mapsto r_\alpha(t)$ from the location of the poles of $\widehat{r}_\alpha(z)$. The arguments can be found in [4]. We only use here that f is C^1, $f'b$ and f^2 are bounded and that $\liminf_{t\to\infty} \frac{1}{t}\int_0^t f(\varphi_s^\alpha(0))\mathrm{d}s > 0$. ☐

We then show that we have convergence in total variation norm:

Lemma 3.7. *There are constants $C, \theta_\alpha > 0$ such that for all $g \in C(\mathbb{R}_+)$ with $\|g\|_\infty < \infty$:*

$$\sup_{x\geq 0} |\mathbb{E}_x g(Y_t^\alpha) - \langle g, \nu_\infty^\alpha \rangle| \leq C\|g\|_\infty e^{-\theta_\alpha t}, \quad \forall t \geq 0.$$

Proof. Recall that $H_\alpha(t)$ is defined by (2.5) and that $K_\alpha(t) := -\frac{\mathrm{d}}{\mathrm{d}t} H_\alpha(t)$. We first show the result for $x = 0$. By Lemma 3.3,

$$\mathbb{E}_0 g(Y_t^\alpha) = g(\varphi_t^\alpha(0)) H_\alpha(t) + \int_0^t K_\alpha(u) \mathbb{E}_0 g(Y_{t-u}^\alpha) \mathrm{d}u.$$

We solve this Volterra equation and find:

$$\mathbb{E}_0 g(Y_t^\alpha) = g(\varphi_t^\alpha(0)) H_\alpha(t) + \int_0^t r_\alpha(t-u) g(\varphi_u^\alpha(0)) H_\alpha(u) \mathrm{d}u.$$

We used here that r_α is the resolvent of K_α, see [15, Ch. 2]. We write $r_\alpha(t) = \gamma(\alpha) + \xi_\alpha(t)$ with $|\xi_\alpha(t)| \leq C e^{-\theta_\alpha t}$. We deduce that

$$\begin{aligned}\mathbb{E}_0 g(Y_t^\alpha) - \gamma(\alpha) \int_0^\infty g(\varphi_u^\alpha(0)) H_\alpha(u) \mathrm{d}u &= g(\varphi_0^\alpha(t)) H_\alpha(t) + \int_0^t \xi_\alpha(t-u) g(\varphi_0^\alpha(u)) H_\alpha(u) \mathrm{d}u \\ &\quad - \gamma(\alpha) \int_t^\infty g(\varphi_0^\alpha(u)) H_\alpha(u) \mathrm{d}u =: A_1 + A_2 + A_3.\end{aligned}$$

Recall that ν_∞^α is given by (2.4). The change of variable $x = \varphi_u^\alpha(0)$ shows that

$$\gamma(\alpha) \int_0^\infty g(\varphi_u^\alpha(0)) H_\alpha(u) \mathrm{d}u = \langle g, \nu_\infty^\alpha \rangle.$$

Using (2.2), we deduce that there exists constants $C, \lambda > 0$ such that

$$\forall t \geq 0, \quad H_\alpha(t) \leq C e^{-\lambda t}.$$

Without loss of generality, we can choose $\theta_\alpha < \lambda/2$. Therefore, $|A_1| \leq C\|g\|_\infty e^{-2\theta_\alpha t}$. Similarly,

$$|A_2| \leq C^2 \|g\|_\infty \int_0^t e^{-\theta_\alpha(t-u)} e^{-2\theta_\alpha u} \mathrm{d}u \leq C^2 \|g\|_\infty e^{-\theta_\alpha t} \int_0^\infty e^{-\theta_\alpha u} \mathrm{d}u.$$

Finally,

$$|A_3| \leq C\|f\|_\infty \|g\|_\infty e^{-\theta_\alpha t} \int_0^\infty e^{-\theta_\alpha u} \mathrm{d}u.$$

Altogether, there exists another constant C such that:

$$|\mathbb{E}_0 g(Y_t^\alpha) - \langle g, \nu_\infty^\alpha \rangle| \leq C\|g\|_\infty e^{-\theta_\alpha t}.$$

Finally, we treat the general case. Recall that $H_\alpha^x(t)$ and $K_\alpha^x(t)$ are defined by (3.2). For all $x \geq 0$, we have by Lemma 3.3:

$$\mathbb{E}_x g(Y_t^\alpha) = g(\varphi_t^\alpha(x)) H_\alpha^x(t) + \int_0^t K_\alpha^x(u) \mathbb{E}_0 g(Y_{t-u}^\alpha) \mathrm{d}u,$$

and so, using that $\int_0^\infty K_\alpha^x(u) \mathrm{d}u = 1$, it holds that

$$\mathbb{E}_x g(Y_t^\alpha) - \langle g, \nu_\infty^\alpha \rangle = g(\varphi_t^\alpha(x)) H_\alpha^x(t) + \int_0^t K_\alpha^x(u) (\mathbb{E}_0 g(Y_{t-u}^\alpha) - \langle g, \nu_\infty^\alpha \rangle) \mathrm{d}u + \langle g, \nu_\infty^\alpha \rangle H_\alpha^x(t).$$

Therefore, the stated estimate is deduced from the case $x = 0$. ∎

Recall that the constant $\underline{\lambda} > 0$ is defined by (2.2). The next step is to prove that

Lemma 3.8. *There exists a constant C such that for all $g \in C^1(\mathbb{R}_+)$ with $\|g\|_\infty \leq 1$ and $\|g'\|_\infty \leq 1$, for all $x, y \geq 0$:*

$$|g(\varphi_t^\alpha(x))H_\alpha^x(t) - g(\varphi_t^\alpha(y))H_\alpha^y(t)| \leq C|x-y|e^{-(\underline{\lambda}/2)t}.$$

Proof. We first show the result when $g \equiv 1$. We use the inequality $|e^{-A} - e^{-B}| \leq e^{-\min(A,B)}|A - B|$, valid for all $A, B \geq 0$. Using Assumption 2.3, we deduce that

$$|H_\alpha^x(t) - H_\alpha^y(t)| \leq Ce^{-(2\underline{\lambda}/3)t} \int_0^t |f(\varphi_u^\alpha(x)) - f(\varphi_u^\alpha(y))| \mathrm{d}u.$$

Using that f is globally Lipschitz and that $|\varphi_t^\alpha(x) - \varphi_t^\alpha(y)| \leq C|x - y|$, we deduce the stated inequality. The general case is deduced similarly, as g and g' are assumed to be bounded by one. □

Finally, we give the proof of Proposition 2.5. In what follows, the constant $\lambda > 0$ might decrease from line to line. Let $g \in C^1(\mathbb{R}_+)$ with $\|g\|_\infty \leq 1$ and $\|g'\|_\infty \leq 1$. We write

$$\mathbb{E}_x g(Y_t^\alpha) - \mathbb{E}_y g(Y_t^\alpha) = g(\varphi_t^\alpha(x))H_\alpha^x(t) - g(\varphi_t^\alpha(y))H_\alpha^y(t) + \int_0^t (K_\alpha^x(u) - K_\alpha^y(u))\mathbb{E}_0 g(Y_{t-u}^\alpha)\mathrm{d}u.$$

The first term is bounded by $C|x - y|e^{-\underline{\lambda}t}$ by the previous lemma. In addition,

$$\forall t \geq u, \quad |\mathbb{E}_0 g(Y_{t-u}^\alpha) - \langle g, \nu_\infty^\alpha \rangle| \leq Ce^{-\lambda(t-u)}.$$

So, using that $\int_0^\infty (K_\alpha^x(u) - K_\alpha^y(u))\mathrm{d}u = 0$, we find that the second term is bounded by

$$|H_\alpha^x(t) - H_\alpha^y(t)||\langle g, \nu_\infty^\alpha \rangle| + C|x - y|e^{-\lambda t}.$$

Altogether, we deduce that there is a constant $C > 0$ and $\lambda > 0$ such that for all x, y:

$$|\mathbb{E}_x g(Y_t^\alpha) - \mathbb{E}_y g(Y_t^\alpha)| \leq C|x - y|e^{-\lambda t}.$$

We define:

$$v_t(x) := (\mathbb{E}_x g(Y_t^\alpha) - \langle g, \nu_\infty^\alpha \rangle)e^{\lambda t}.$$

In view of Lemma 3.3, it holds that $v_t \in C^1(\mathbb{R}_+)$. By the previous results, we have for some constant C (independent of g):

$$\|v_t\|_\infty + \|v_t'\|_\infty \leq C.$$

So

$$|\langle v_t, \nu - \mu \rangle| \leq C\|\nu - \mu\|_0.$$

In other words, by the Markov property:

$$\|Law(Y_t^{\alpha,\nu}) - Law(Y_t^{\alpha,\mu})\|_0 \leq Ce^{-\lambda t}\|\nu - \mu\|_0.$$

This ends the proof. □

3.4 Reformulation of the spectral assumption

In this section, we reformulate the spectral assumption (2.7). We recall that

$$H^y_\alpha(t) = \exp\left(-\int_0^t f(\varphi^\alpha_u(y))\mathrm{d}u\right), \quad H_\alpha(t) = H^0_\alpha(t),$$
$$\Psi_\alpha(t) := \alpha \int_0^\infty H_\alpha(t+u) \frac{f(\varphi^\alpha_{t+u}(0)) - f(\varphi^\alpha_u(0))}{b(\varphi^\alpha_u(0)) + \alpha}\mathrm{d}u.$$

The first step is to show that

Lemma 3.9. *It holds that*

$$\Psi_\alpha(t) = -J\int_0^{\sigma_\alpha} \left[\frac{\mathrm{d}}{\mathrm{d}y} H^y_\alpha(t)\right] \nu^\alpha_\infty(\mathrm{d}y). \tag{3.3}$$

Proof. First, note that the function $y \mapsto \varphi^\alpha_u(y)$ is $C^1(\mathbb{R}_+)$ with

$$\frac{\mathrm{d}}{\mathrm{d}y}\varphi^\alpha_u(y) = \frac{b(\varphi^\alpha_u(y)) + \alpha}{b(y) + \alpha}. \tag{3.4}$$

Indeed, both the left-hand-side and the right-and-side of (3.4) satisfies the ODE: $\partial_u \psi_u = b'(\varphi^\alpha_u(y))\psi_u$, with $\psi_0 = 1$. By uniqueness, (3.4) follows.

Therefore, the function $y \mapsto H^y_\alpha(t)$ is C^1 with

$$\frac{\mathrm{d}}{\mathrm{d}y} H^y_\alpha(t) = -H^y_\alpha(t)\int_0^t f'(\varphi^\alpha_u(y))\frac{\mathrm{d}}{\mathrm{d}y}\varphi^\alpha_u(y)\mathrm{d}u = -H^y_\alpha(t)\frac{f(\varphi^\alpha_t(y)) - f(y)}{b(y)+\alpha}.$$

We used (3.4) to obtain the last equality. Let $A(t)$ be equal to the right-end side of (3.3). Plugging the explicit expression of ν^∞_α (see (2.4)) and using that $J = \alpha/\gamma(\alpha)$, we find

$$\begin{aligned} A(t) &= \alpha\int_0^{\sigma_\alpha} H^x_\alpha(t)\frac{f(\varphi^\alpha_t(x)) - f(x)}{(b(x)+\alpha)^2}\exp\left(-\int_0^x \frac{f(y)}{b(y)+\alpha}\mathrm{d}y\right)\mathrm{d}x \\ &= \alpha\int_0^\infty \exp\left(-\int_0^t f(\varphi^\alpha_{s+u}(0))\mathrm{d}s\right)\frac{f(\varphi^\alpha_{t+u}(0)) - f(\varphi^\alpha_u(0))}{b(\varphi^\alpha_u(0))+\alpha} H_\alpha(u)\mathrm{d}u. \end{aligned}$$

To obtain the last equality we made the change of variables $x = \varphi^\alpha_u(0)$ and then $y = \varphi^\alpha_\theta(0)$. Hence, we find that $A(t) = \Psi_\alpha(t)$ as claimed. ☐

Then, we define

$$\Theta_\alpha(t) := J\int_{\mathbb{R}_+}\frac{\mathrm{d}}{\mathrm{d}y}\mathbb{E}_y f(Y^\alpha_t)\nu^\alpha_\infty(\mathrm{d}y).$$

Recall that $\lambda_\alpha > 0$ is defined in Assumption 2.6 and that $D_\alpha := \{z \in \mathbb{C};\ \Re(z) > -\lambda_\alpha\}$.

Lemma 3.10. *For all $z \in D_\alpha$, it holds that $\hat{\Psi}_\alpha(z) = \hat{H}_\alpha(z)$ if and only if $\hat{\Theta}_\alpha(z) = 1$.*

Proof. Recall that $r^x_\alpha(t) := \mathbb{E}_x f(Y^\alpha_t)$. Using Lemma 3.2, we have

$$r^x_\alpha = K^x_\alpha + r_\alpha * K^x_\alpha.$$

We used here the notation $(r_\alpha * K^x_\alpha)(t) := \int_0^t r_\alpha(t-s)K^x_\alpha(s)\mathrm{d}s$. We differentiate with respect to x and obtain

$$J\frac{\mathrm{d}}{\mathrm{d}x}r^x_\alpha = J\frac{\mathrm{d}}{\mathrm{d}x}K^x_\alpha + r_\alpha * \left[J\frac{\mathrm{d}}{\mathrm{d}x}K^x_\alpha\right].$$

Let

$$\Xi_\alpha(t) := \frac{\mathrm{d}}{\mathrm{d}t}\Psi_\alpha(t). \tag{3.5}$$

In view of (3.3) and $\frac{\mathrm{d}}{\mathrm{d}t}H_\alpha^x(t) = -K_\alpha^x(t)$, we have:

$$\Xi_\alpha(t) = J \int_{\mathbb{R}_+} \frac{\mathrm{d}}{\mathrm{d}y} K_\alpha^y(t) \nu_\infty^\alpha(\mathrm{d}y).$$

We deduce that

$$\Theta_\alpha = \Xi_\alpha + r_\alpha * \Xi_\alpha.$$

Taking the Laplace transform, we find that for all $z \in \mathbb{C}$ with $\Re(z) > 0$:

$$\widehat{\Theta}_\alpha(z) = \widehat{\Xi}_\alpha(z) + \widehat{r}_\alpha(z)\widehat{\Xi}_\alpha(z).$$

Therefore, we have

$$\begin{aligned} \widehat{\Theta}_\alpha(z) &= \widehat{\Xi}_\alpha(z)\left[1 + \widehat{r}_\alpha(z)\right] \\ &= \widehat{\Xi}_\alpha(z)\left[1 + \frac{\widehat{K}_\alpha(z)}{1 - \widehat{K}_\alpha(z)}\right] \quad (\text{using } r_\alpha = K_\alpha + K_\alpha * r_\alpha) \\ &= \frac{\widehat{\Xi}_\alpha(z)}{z\widehat{H}_\alpha(z)} \quad (\text{using } \widehat{K}_\alpha(z) = 1 - z\widehat{H}_\alpha(z)) \\ &= \frac{\widehat{\Psi}_\alpha(z)}{\widehat{H}_\alpha(z)} \quad (\text{using } \Psi_\alpha(0) = 0 \text{ and } \frac{\mathrm{d}}{\mathrm{d}t}\Psi_\alpha = \Xi_\alpha). \end{aligned} \tag{3.6}$$

Because the left-hand side and the right-hand side are two analytic functions on D_α, the equality is in fact valid on D_α and so

$$\forall z \in D_\alpha, \quad \widehat{\Theta}_\alpha(z) = 1 \iff \widehat{\Psi}_\alpha(z) = \widehat{H}_\alpha(z).$$

□

We finally consider $\Omega_\alpha(t)$ the solution of the Volterra integral equation

$$\forall t \geq 0, \quad \Omega_\alpha(t) = \Theta_\alpha(t) + \int_0^t \Omega_\alpha(t-s)\Theta_\alpha(s)\mathrm{d}s.$$

Remark 3.11. The function $\Omega_\alpha(t)$ has a simple probabilistic interpretation using the McKean-Vlasov equation (1.1). For all $\epsilon > 0$, let $\nu_\epsilon := Law(X_0^{\nu_\infty} + \epsilon)$, with $Law(X_0^{\nu_\infty}) = \nu_\infty$. Then

$$\Omega_\alpha(t) = \lim_{\epsilon \downarrow 0} \frac{\mathbb{E}f(X_t^{\nu_\epsilon}) - \mathbb{E}f(X_t^{\nu_\infty})}{\epsilon}.$$

Similarly, it holds that

$$\Theta_\alpha(t) = \lim_{\epsilon \downarrow 0} \frac{\mathbb{E}f(Y_t^{\alpha,\nu_\epsilon}) - \mathbb{E}f(Y_t^{\alpha,\nu_\infty})}{\epsilon},$$

where $(Y_t^{\alpha,\nu})$ is the solution of (2.1). We refer to [3] for these probabilistic interpretations as well as the connection with Lions derivatives.

Lemma 3.12. *For all $\lambda < \lambda'_\alpha$, where λ'_α is given by (2.7), we have $\sup_{t\geq 0}|\Omega_\alpha(t)|e^{\lambda t} < \infty$.*

In other words, λ'_α gives the rate of convergence of $\Omega_\alpha(t)$ towards zero.

Proof. Let $\lambda < \lambda'$, $K_t := e^{\lambda t}\Theta_\alpha(t)$ and $R_t := e^{\lambda t}\Omega_\alpha(t)$. It holds that $K \in L^1(\mathbb{R}_+)$. By assumption, it holds that $\hat{K}(z) \neq 0$ for all $\Re(z) \geq 0$. Therefore, [15, Ch. 2, Th. 4.1] applies and so $R \in L^1(\mathbb{R}_+)$. Finally, using that $R = K + K * R$ we find that R is also bounded. □

3.5 Proof of Theorem 2.8

A sensitivity formula

Following [3], we first show the following "sensitivity" formula:

Proposition 3.13. *Let* $k \in C([0,t];\mathbb{R})$ *and* $\alpha \in \mathbb{R}_+$*, such that* $\inf_{s\in[0,t]}(\alpha + k_s) \geq 0$*. Provided that* $Law(Y_0^{\alpha+k}) = Law(Y_0^{\alpha})$*, it holds that*

$$\mathbb{E}g(Y_t^{\alpha+k}) - \mathbb{E}g(Y_t^{\alpha}) = \int_0^t \int_{\mathbb{R}_+} \left[\frac{\mathrm{d}}{\mathrm{d}y}\mathbb{E}_y g(Y_{t-\theta}^{\alpha})\right] k_\theta Law(Y_\theta^{\alpha+k})(\mathrm{d}y)\mathrm{d}\theta.$$

Proof. This is a Trotter-Kato formula. Define for all $s \in (0,t]$ and for all $y \in \mathbb{R}_+$:

$$\phi(s,y) := \mathbb{E}_y g(Y_{t-s}^{\alpha}).$$

The function ϕ is $C_b^1(\mathbb{R}_+ \times \mathbb{R}_+)$ and

$$\frac{\partial}{\partial s}\phi(s,y) = -\mathcal{L}^{\alpha}\phi(s,y),$$

where $\mathcal{L}^{\alpha}$ is the generator of (Y_t^{α}), solution of (2.1). This generator acts on the marginal function $\phi(s,\cdot)$ and is given by

$$\forall g \in C_b^1(\mathbb{R}_+), \quad \mathcal{L}^{\alpha}g(y) := g'(y)(b(y)+\alpha) + (g(0)-g(y))f(y).$$

In addition, the generator of $(Y_s^{\alpha+k})$ satisfies $(\mathcal{L}_s^{\alpha+k} - \mathcal{L}^{\alpha})g(y) = g'(y)k_s$. Therefore, by Ito's formula, we obtain:

$$\mathbb{E}\phi(s, Y_s^{\alpha+k}) = \mathbb{E}\phi(0, Y_0^{\alpha+k}) + \mathbb{E}\int_0^s \frac{\partial}{\partial y}\phi(u, Y_u^{\alpha+k})k_u \mathrm{d}u.$$

Replacing ϕ by its definition, we deduce that

$$\mathbb{E}\phi(s, Y_s^{\alpha+k}) = \mathbb{E}\phi(0, Y_0^{\alpha+k}) + \mathbb{E}\int_0^s \int_{\mathbb{R}_+}\left[\frac{\mathrm{d}}{\mathrm{d}y}\mathbb{E}_y g(Y_{t-u}^{\alpha})\right] k_u Law(Y_u^{\alpha+k})(\mathrm{d}y)\mathrm{d}u.$$

Finally, we let s converge to t. Using the Markov property at time $s=0$ and the fact that $\phi(t,y) = g(y)$, we find that the stated formula holds. ◻

Corollary 3.14. *It holds that*

$$\|Law(Y_t^{\alpha+k,\nu}) - Law(Y_t^{\alpha,\nu})\|_0 \leq C_* \int_0^t e^{-\lambda_*(t-s)}|k_s|\mathrm{d}s.$$

Proof. Let $g \in Lip_1(\mathbb{R}_+)$ with $\|g\|_\infty \leq 1$. We have by Proposition 3.13:

$$|\mathbb{E}g(Y_t^{\alpha+k,\nu}) - \mathbb{E}g(Y_t^{\alpha,\nu})| \leq \int_0^t \left\{\sup_{y\in\mathbb{R}_+}\left|\frac{\mathrm{d}}{\mathrm{d}y}\mathbb{E}_y g(Y_{t-s}^{\alpha})\right|\right\}|k_s|\mathrm{d}s.$$

By Proposition 2.5, we have for $y \neq y'$:

$$|\mathbb{E}_y g(Y_t^{\alpha}) - \mathbb{E}_{y'}g(Y_t^{\alpha})| \leq C_* e^{-\lambda_* t}\|\delta_y - \delta_{y'}\|_0.$$

As $\|\delta_y - \delta_{y'}\|_0 = |y-y'|$ for $|y-y'| \leq 1$, we deduce that $\left|\frac{\mathrm{d}}{\mathrm{d}y}\mathbb{E}_y g(Y_{t-s}^{\alpha})\right| \leq C_* e^{-\lambda_*(t-s)}$. ◻

Control of the non-linear interactions

We define:

$$\varphi_t^\nu := J\mathbb{E}f(Y_t^{\alpha,\nu}) - \alpha$$
$$k_t^\nu = J\mathbb{E}f(X_t^\nu) - \alpha.$$

We prove that:

Proposition 3.15. *For all $T > 0$, there is a constant C_T such that for all $t \in [0,T]$ and for all $\nu \in \mathcal{P}(\mathbb{R}^d)$:*

1. $|k_t^\nu| \leq C_T \|\nu - \nu_\infty\|_0$.
2. $\left| k_t^\nu - \varphi_t^\nu - \int_0^t \Theta_\alpha(t-s) k_s^\nu \mathrm{d}s \right| \leq C_T \left(\|\nu - \nu_\infty\|_0 \right)^2$.
3. $\left| k_t^\nu - \varphi_t^\nu - \int_0^t \Omega_\alpha(t-s) \varphi_s^\nu \mathrm{d}s \right| \leq C_T \left(\|\nu - \nu_\infty\|_0 \right)^2$.

Proof. The first point is a consequence of $\|f\|_\infty + \|f'\|_\infty < \infty$ and of Theorem 2.2. For the second point, we note that $\mathbb{E}f(X_t^\nu) = \mathbb{E}f(Y_t^{\alpha+k^\nu,\nu})$. We define $\psi_t(y) = \frac{\mathrm{d}}{\mathrm{d}y}\mathbb{E}_y f(Y_t^\alpha)$. We have, using Proposition 3.13 with $g = Jf$:

$$\begin{aligned}
k_t^\nu - \varphi_t^\nu &= J\mathbb{E}f(Y_t^{\alpha+k^\nu,\nu}) - J\mathbb{E}f(Y_t^{\alpha,\nu}) \\
&= J\int_0^t \int_{\mathbb{R}_+} \frac{\mathrm{d}}{\mathrm{d}y}\mathbb{E}_y f(Y_{t-s}^\alpha) k_s^\nu Law(Y_s^{\alpha+k^\nu,\nu})(\mathrm{d}y)\mathrm{d}s \\
&= J\int_0^t \mathbb{E}\psi_{t-s}(Y_s^{\alpha+k^\nu,\nu}) k_s^\nu \mathrm{d}s \\
&= \int_0^t \Theta_\alpha(t-s) k_s^\nu \mathrm{d}s + J\int_0^t \left[\mathbb{E}\psi_{t-s}(Y_s^{\alpha+k^\nu,\nu}) - \mathbb{E}\psi_{t-s}(Y_s^{\alpha,\nu_\infty})\right] k_s^\nu \mathrm{d}s.
\end{aligned}$$

Because f and b are assumed to be C^2, there exists a constant C_T such that

$$\forall t \in [0,T], \quad \|\psi_t\|_\infty + \|\partial_y \psi_t\|_\infty \leq C_T.$$

Therefore, by Theorem 2.2 we have:

$$\left|\mathbb{E}\psi_{t-s}(Y_s^{\alpha+k^\nu,\nu}) - \mathbb{E}\psi_{t-s}(Y_s^{\alpha,\nu_\infty})\right| = |\mathbb{E}\psi_{t-s}(X_s^\nu) - \mathbb{E}\psi_{t-s}(X_s^{\nu_\infty})| \leq C_T\|\nu - \nu_\infty\|_0.$$

Using the first point, we obtain the stated inequality. The last point is obtained by iterating the estimate of the second point, as in [3]. ☐

Exactly as in [3, Lem. 2.20], we deduce from this last result and from Corollary 3.14 that:

Lemma 3.16. *Let $\lambda \in (0, \lambda'_\alpha)$. There exists a constant C_λ such that for all $T > 0$, there exists $C_T > 0$: for all $\nu \in \mathcal{P}(\mathbb{R})$, for all $t \in [0,T]$,*

$$\|Law(X_t^\nu) - \nu_\infty\|_0 \leq C_\lambda e^{-\lambda t}\|\nu - \nu_\infty\|_0 + C_T\left(\|\nu - \nu_\infty\|_0\right)^2.$$

We used crucially here that $\sup_{t\geq 0}|\Omega_\alpha(t)|e^{\lambda t} < \infty$, see Lemma 3.12. The proof of Theorem 2.8 is easily deduced from Lemma 3.16, exactly as in [3].

3.6 Proof of Proposition 2.9

The first point is to note that under the assumption $\inf_{[0,\sigma_\alpha)} f + b' \geq 0$, we can integrate by parts Ψ_α and Ξ_α:

Lemma 3.17. *We have*

1. *The following limit exists and is finite:* $\nu^\infty_\alpha(\sigma_\alpha) := \lim_{x\uparrow\sigma_\alpha} \nu^\infty_\alpha(x) < \infty$.
2. *Define* $C_\alpha := \frac{b(0)+\alpha}{\gamma(\alpha)}\nu^\infty_\alpha(\sigma_\alpha)$ *and*

$$\Upsilon_\alpha(t) := C_\alpha H^{\sigma_\alpha}_\alpha(t) + \int_0^\infty H_\alpha(t+u)\left[f(\varphi^\alpha_u(0)) + b'(\varphi^\alpha_u(0))\right]\frac{b(0)+\alpha}{b(\varphi^\alpha_u(0))+\alpha}\mathrm{d}u. \quad (3.7)$$

It holds that for all $t \geq 0$*:*

$$\Psi_\alpha(t) = \frac{\alpha}{b(0)+\alpha}\left[H_\alpha(t) - \Upsilon_\alpha(t)\right]. \quad (3.8)$$

3. *Define* $\Lambda_\alpha(t) := -\frac{\mathrm{d}}{\mathrm{d}t}\Upsilon_\alpha(t)$. *One has for all* $t \geq 0$

$$\Lambda_\alpha(t) = C_\alpha K^{\sigma_\alpha}_\alpha(t) + \int_0^\infty K_\alpha(t+u)\left[f(\varphi^\alpha_u(0)) + b'(\varphi^\alpha_u(0))\right]\frac{b(0)+\alpha}{b(\varphi^\alpha_u(0))+\alpha}\mathrm{d}u. \quad (3.9)$$

Moreover, for all $t \geq 0$

$$\Xi_\alpha(t) = \frac{\alpha}{b(0)+\alpha}\left[\Lambda_\alpha(t) - K_\alpha(t)\right]. \quad (3.10)$$

Proof of Lemma 3.17. To prove the first point, we use the explicit formula of the invariant measure (2.4): we find that for all $x < \sigma_\alpha$

$$\frac{\mathrm{d}}{\mathrm{d}x}\nu^\alpha_\infty(x) = -\gamma(\alpha)\frac{f(x)+b'(x)}{(b(x)+\alpha)^2}\exp\left(-\int_0^x \frac{f(y)}{b(y)+\alpha}\mathrm{d}y\right) \leq 0.$$

Therefore, $x \mapsto \nu^\alpha_\infty(x)$ is non-increasing and so $\lim_{x\uparrow\sigma_\alpha}\nu^\alpha_\infty(x)$ exists and is finite (it is equals to zero if $\sigma_\alpha = \infty$, and might be non-null in the case where $\sigma_\alpha < \infty$). To prove the second point, we integrate by parts the right-hand side of (3.3) and find

$$\Psi_\alpha(t) = \frac{\alpha}{b(0)+\alpha}\left[H_\alpha(t) - C_\alpha H^{\sigma_\alpha}_\alpha(t)\right] + J\int_0^{\sigma_\alpha} H^x_\alpha(t)\frac{\mathrm{d}}{\mathrm{d}x}\nu^\alpha_\infty(x)\mathrm{d}x.$$

The last term is equal to:

$$J\int_0^{\sigma_\alpha} H^x_\alpha(t)\frac{\mathrm{d}}{\mathrm{d}x}\nu^\alpha_\infty(x)\mathrm{d}x = -\alpha\int_0^{\sigma_\alpha} H^x_\alpha(t)\frac{f(x)+b'(x)}{(b(x)+\alpha)^2}\exp\left(-\int_0^x \frac{f(y)}{b(y)+\alpha}\mathrm{d}y\right)\mathrm{d}y.$$

We make the changes of variables $y = \varphi^\alpha_\theta(0)$ and $x = \varphi^\alpha_u(0)$ and obtain

$$\Psi_\alpha(t) = \frac{\alpha}{b(0)+\alpha}\left[H_\alpha(t) - C_\alpha H^{\sigma_\alpha}_\alpha(t)\right] - \alpha\int_0^\infty H^{\varphi^\alpha_u(0)}_\alpha(t)\frac{f(\varphi^\alpha_u(0)) + b'(\varphi^\alpha_u(0))}{b(\varphi^\alpha_u(0))+\alpha}H_\alpha(u)\mathrm{d}u.$$

Using that $H^{\varphi^\alpha_u(0)}_\alpha(t)H_\alpha(u) = H_\alpha(t+u)$, we obtain the stated formula. Finally, recall that $\Xi_\alpha(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Psi_\alpha(t)$. Therefore, the third point is obtained by differentiating the second point with respect to t. ◻

Proof of Proposition 2.9, first point. Recall that $\Xi_\alpha(t)$ is given by (3.5) and satisfies for all $z \in D_\alpha$, $\widehat{\Xi}_\alpha(z) = z\widehat{\Psi}_\alpha(z)$. Similarly, it holds that $\widehat{K}_\alpha(z) = 1 - z\widehat{H}_\alpha(z)$. Let $z_* \in D_\alpha$ such that $\widehat{H}_\alpha(z_*) = \widehat{\Psi}_\alpha(z_*)$. We deduce that

$$1 - \widehat{K}_\alpha(z_*) = \widehat{\Xi}_\alpha(z_*).$$

Using Lemma 3.17, we have

$$b(0) + \alpha = b(0)\widehat{K}_\alpha(z_*) + \alpha\widehat{\Lambda}_\alpha(z_*). \tag{3.11}$$

We show that $\Re(z_*) < 0$. Indeed, we have

$$\Re(z) > 0 \implies |b(0)\widehat{K}_\alpha(z) + \alpha\widehat{\Lambda}_\alpha(z)| < b(0)|\widehat{K}_\alpha(z)| + \alpha|\widehat{\Lambda}_\alpha(z)| < b(0) + \alpha,$$

and so necessarily, $\Re(z_*) \leq 0$. We used here that both $K_\alpha(t)$ and $\Lambda_\alpha(t)$ are the densities of probability measures. In addition, if $z_* = i\omega$ for some $\omega > 0$, then

$$\Re\left[b(0)(1 - \widehat{K}_\alpha(i\omega)) + \alpha(1 - \widehat{\Lambda}_\alpha(i\omega))\right] = \int_0^\infty [1 - \cos(\omega t)](b(0)K_\alpha(t) + \alpha\Lambda_\alpha(t))\mathrm{d}t.$$

Because for $t \in \mathbb{R}_+$, $1 - \cos(\omega t) > 0$ almost everywhere, the right-hand side is null only if almost everywhere

$$b(0)K_\alpha(t) + \alpha\Lambda_\alpha(t) = 0.$$

This leads to a contradiction because by (3.9), we have $\Lambda_\alpha(t) \geq 0$. In addition, $K_\alpha(t) \geq 0$ and the total mass of K_α is one. Altogether, we have proved that

$$\forall z_* \in D_\alpha, \quad \widehat{H}_\alpha(z_*) = \widehat{\Psi}_\alpha(z_*) \implies \Re(z_*) < 0.$$

Using the Riemann–Lebesgue lemma, we deduce that $\lambda'_\alpha > 0$. □

Proof of Proposition 2.9, second point. Assume that $\inf_{x\geq 0} f(x) + b'(x) \geq 0$ and $b(0) > 0$. The number of invariant probability measures of (1.1) is given by the number of solutions of the equation $\alpha = J\gamma(\alpha), \alpha \geq 0$. We first prove that the continuous function $G : \alpha \mapsto \frac{\alpha}{\gamma(\alpha)}$ is strictly increasing on $\mathbb{R}_+$. Note first the identity:

$$\forall t \geq 0, \quad [b(\varphi_t^\alpha(0)) + \alpha]\exp\left(-\int_0^t b'(\varphi_u^\alpha(0))\mathrm{d}u\right) = b(0) + \alpha.$$

We deduce that for all $\alpha > 0$

$$\begin{aligned}
G(\alpha) = \frac{\alpha}{\gamma(\alpha)} &= \alpha\int_0^\infty H_\alpha(t)\mathrm{d}t \\
&= \frac{\alpha}{b(0)+\alpha}\int_0^\infty [b(\varphi_t^\alpha(0)) + \alpha]\exp\left(-\int_0^t b'(\varphi_u^\alpha(0))\mathrm{d}u\right)H_\alpha(t)\mathrm{d}t \\
&= \frac{\alpha}{b(0)+\alpha}\int_0^\infty [b(\varphi_t^\alpha(0)) + \alpha]\exp\left(-\int_0^t (f+b')(\varphi_u^\alpha(0))\mathrm{d}u\right)\mathrm{d}t.
\end{aligned}$$

The changes of variables $y = \varphi_u^\alpha(0)$ and $x = \varphi_t^\alpha(0)$ show that

$$\frac{\alpha}{\gamma(\alpha)} = \frac{\alpha}{b(0)+\alpha}\int_0^{\sigma_\alpha}\exp\left(-\int_0^x \frac{(f+b')(y)}{b(y)+\alpha}\mathrm{d}y\right)\mathrm{d}x.$$

Note that the function $\alpha \mapsto \frac{\alpha}{b(0)+\alpha}$ is non-decreasing and $\alpha \mapsto \sigma_\alpha$ is strictly increasing. Moreover, because $f + b' \geq 0$, for all fixed x, the function

$$\alpha \mapsto \exp\left(-\int_0^x \frac{(f+b')(y)}{b(y)+\alpha}\mathrm{d}y\right)$$

is non-decreasing. So G is strictly increasing. Because $b(0) > 0$, we have $\gamma(0) > 0$ and so $G(0) = 0$. Therefore, for all $J \geq 0$, the equation $G(\alpha) = J$ has a unique solution. □

4 An illustrated example

To illustrate the results, we consider

$$f(x) = x^2 \quad \text{and} \quad b(x) = -x, \quad \forall x \geq 0.$$

There is a slight technical difficulty: f and f' are not bounded and so we cannot directly apply our results. For $A > 0$, we denote by

$$\mathcal{M}_A := \{\nu \in \mathcal{P}(\mathbb{R}_+); \quad J\langle f, \nu\rangle \leq A, \quad \nu([0, A]) = 1\}.$$

Lemma 4.1. *Let $J \geq 0$. There exists a constant $A > 0$ large enough such that for any initial condition $\nu \in \mathcal{M}_A$, there is a unique path-wise solution to (1.1) and for all $t \geq 0$, $Law(X_t^\nu) \in \mathcal{M}_A$.*

Proof. Existence and uniqueness of the solution of (1.1) is not problematic since the initial condition is compactly supported, see [12]. The existence of the constant A is shown in [4]. The idea is that by Ito's formula,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}f(X_t^\nu) &= \mathbb{E}f'(X_t^\nu)[b(X_t^\nu) + J\mathbb{E}f(X_t^\nu)] - \mathbb{E}f^2(X_t^\nu) \\
&\leq \frac{C^2}{2} - \frac{1}{2}\mathbb{E}f^2(X_t^\nu) \\
&\leq \frac{C^2}{2} - \frac{1}{2}(\mathbb{E}f(X_t^\nu))^2,
\end{aligned}$$

for some constant C only depending on J. We used the Cauchy-Schwarz inequality to obtain the last estimate. Then we deduce that $\mathbb{E}f(X_0^\nu) \leq C$ implies $\mathbb{E}f(X_t^\nu) \leq C$ for all $t \geq 0$. Therefore, we can choose $A = JC$ to obtain the result. □

We equip $\mathcal{M}_A$ with the distance $\|\nu - \mu\|_0$ (we could also use here the standard Wasserstein distance $W_1(\nu, \mu)$: W_1 and $\|\cdot\|_0$ are equivalent on $\mathcal{M}_A$, as the probability measures are compactly supported). In view of Lemma 4.1, as soon as the initial condition belongs to $\mathcal{M}_A$, everything happens as if f and f' were bounded on $\mathbb{R}_+$. Therefore Theorem 2.2 and 2.8 holds with $\mathcal{P}(\mathbb{R}_+)$ being replaced with $\mathcal{M}_A$.

Our goal is now to describe all the invariant probability measures and to find their stability properties. The first step to apply Theorem 2.8 is to specify the values of $\widehat{H}_\alpha$ and $\widehat{\Psi}_\alpha$.

Computation of $\widehat{H_\alpha}(z)$ and $\widehat{\Psi_\alpha}(z)$

Let $\alpha > 0$. Recall that $H_\alpha(t)$ is defined by (2.5) and that $\Psi_\alpha(t)$ is defined by (2.6). Let:

$$\forall \theta \in [0, 1), \quad w(\theta) := \theta + \frac{\theta^2}{2} + \log(1 - \theta) = -\sum_{k \geq 3} \frac{\theta^k}{k}.$$

In this section, we prove that:

Proposition 4.2. *Let $\psi(z, x) := \frac{2 - 2(1-x)^z - 2xz - (1-x)^z x^2 (1-z) z}{(1-x)(1-z)z(1+z)}$. It holds that for all $\Re(z) > -\alpha^2$:*

$$\widehat{H_\alpha}(z) = \int_0^1 (1 - x)^{z-1} e^{\alpha^2 w(x)} \mathrm{d}x$$

and

$$\widehat{\Psi_\alpha}(z) = \alpha^2 \int_0^1 \psi(z, x) e^{\alpha^2 w(x)} \mathrm{d}x.$$

To proceed, we introduce some notations. First, recall that $K_\alpha(t) = -\frac{\mathrm{d}}{\mathrm{d}t}H_\alpha(t)$. We also consider

$$\forall \theta \in [0,1), \quad \widehat{H_\alpha^{[\theta]}}(z) := \int_0^\infty e^{-zt} H_\alpha(t - \log(1-\theta))\mathrm{d}t,$$
$$\widehat{K_\alpha^{[\theta]}}(z) := \int_0^\infty e^{-zt} K_\alpha(t - \log(1-\theta))\mathrm{d}t.$$

Lemma 4.3. *The following identities hold:*

$$\widehat{H_\alpha^{[\theta]}}(z) = (1-\theta)^{-z} \int_\theta^1 (1-x)^{z-1} e^{\alpha^2 w(x)}\mathrm{d}x,$$
$$\widehat{K_\alpha^{[\theta]}}(z) = e^{\alpha^2 w(\theta)} - z\widehat{H_\alpha^{[\theta]}}(z).$$

In addition, we have

$$e^{\alpha^2 w(\theta)} = (z+\alpha^2)\widehat{H_\alpha^{[\theta]}}(z) - \alpha^2 \int_\theta^1 \left(\frac{1-x}{1-\theta}\right)^z (1+x)e^{\alpha^2 w(x)}\mathrm{d}x.$$

Proof. We have $\varphi_t^\alpha(0) = \alpha(1-e^{-t})$ and so

$$H_\alpha(t) = \exp\left(-\int_0^t \alpha^2(1-e^{-u})^2\mathrm{d}u\right) = \exp\left(\alpha^2[\frac{3}{2} + \frac{e^{-2t}}{2} - 2e^{-t} - t]\right).$$

Therefore, we find that

$$\forall x \in [0,1), \quad H_\alpha(-\log(1-x)) = e^{\alpha^2 w(x)}.$$

So, the change of variable $x = 1 - e^{-t}$ shows that

$$\widehat{H_\alpha^{[\theta]}}(z) = (1-\theta)^{-z}\int_{-\log(1-\theta)}^\infty e^{-zt}H_\alpha(t)\mathrm{d}t$$
$$= (1-\theta)^{-z}\int_\theta^1 (1-x)^{z-1} e^{\alpha^2 w(x)}\mathrm{d}x.$$

This proves the first equality. The second equality is obtained by an integration by parts, using that $\frac{\mathrm{d}}{\mathrm{d}t}H_\alpha(t-\log(1-\theta)) = -K_\alpha(t-\log(1-\theta))$. To obtain the last identity, we note that:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x)^z e^{\alpha^2 w(x)}\right] = (z+\alpha^2)(1-x)^{z-1}e^{\alpha^2 w(x)} - \alpha^2(1-x)^z(1+x)e^{\alpha^2 w(x)}.$$

We then integrate this equality from $x = \theta$ to $x = 1$. This gives the result. ☐

Using this Lemma, we can finally deduce the expression of $\widehat{\Psi_\alpha}$:

Proof of Proposition 4.2. For $u \in \mathbb{R}_+$, let $\theta(u) := 1 - e^{-u}$. Recall that $\Psi_\alpha(t)$ is given by (2.6). Therefore, it holds that

$$\widehat{\Psi_\alpha}(z) = \alpha\int_0^\infty \frac{1}{b(\varphi_u^\alpha(0))+\alpha}\left[\int_0^\infty e^{-zt}K_\alpha(t+u)\mathrm{d}t - f(\varphi_u^\alpha(0))\int_0^\infty e^{-zt}H_\alpha(t+u)\mathrm{d}t\right]\mathrm{d}u$$
$$= \int_0^\infty e^u\left[\widehat{K_\alpha^{[\theta(u)]}}(z) - \alpha^2(\theta(u))^2\widehat{H_\alpha^{[\theta(u)]}}(z)\right]\mathrm{d}u$$
$$= \int_0^1 (1-\theta)^{-2}\left[\widehat{K_\alpha^{[\theta]}}(z) - \alpha^2\theta^2\widehat{H_\alpha^{[\theta]}}(z)\right]\mathrm{d}\theta.$$

We made the change of variable $\theta = 1 - e^{-u}$. We now use Lemma 4.3 and find that

$$\widehat{K_\alpha^{[\theta]}}(z) - \alpha^2\theta^2\widehat{H_\alpha^{[\theta]}}(z) = \alpha^2(1-\theta^2)\widehat{H_\alpha^{[\theta]}}(z) - \alpha^2 \int_\theta^1 \left(\frac{1-x}{1-\theta}\right)^z (1+x)e^{\alpha^2 w(x)}\mathrm{d}x.$$
$$= \int_\theta^1 \left[\alpha^2(1-\theta^2)\frac{(1-x)^{z-1}}{(1-\theta)^z} - \alpha^2\left(\frac{1-x}{1-\theta}\right)^z (1+x)\right] e^{\alpha^2 w(x)}\mathrm{d}x.$$

Therefore, we find that

$$\widehat{\Psi_\alpha}(z) = \alpha^2 \int_0^1 \frac{1-\theta^2}{(1-\theta)^z} \int_\theta^1 (1-x)^{z-1} e^{\alpha^2 w(x)}\mathrm{d}x\mathrm{d}\theta$$
$$- \alpha^2 \int_0^1 \frac{1}{(1-\theta)^z} \int_\theta^1 (1-x)^z (1+x) e^{\alpha^2 w(x)}\mathrm{d}x\mathrm{d}\theta.$$

To obtain the stated result, it suffices to integrate by parts this equality. □

4.1 Description of the invariant probability measures

The following proposition gives the number of invariant measures of the non-linear equation (1.1). This result is conjectured to be true in [20, Section 7.2.3].

Graph of the function $\alpha \mapsto J(\alpha) := \frac{\alpha}{\gamma(\alpha)}$.

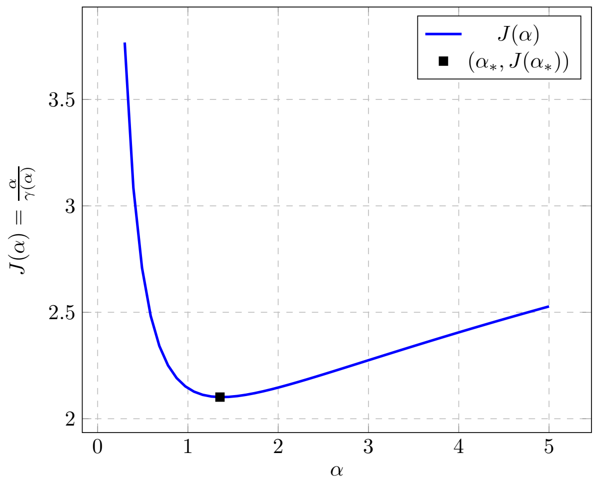


Figure 1: Plot of the function $\alpha \mapsto J(\alpha) := \frac{\alpha}{\gamma(\alpha)}$, for $b(x) = -x$ and $f(x) = x^2$. We prove in Proposition 4.4 that this function is decreasing on $(0, \alpha_*]$ and increasing on $[\alpha_*, \infty)$.

Proposition 4.4. *Let $f(x) = x^2$ and $b(x) = -x$. There exists $\alpha_* > 0$ such that the function $\alpha \mapsto \frac{\alpha}{\gamma(\alpha)}$ is decreasing on $(0, \alpha_*]$ and increasing on $[\alpha_*, \infty)$. Moreover, one has*

$$\lim_{\alpha \downarrow 0} \frac{\alpha}{\gamma(\alpha)} = +\infty, \quad \textit{and} \quad \lim_{\alpha \to \infty} \frac{\alpha}{\gamma(\alpha)} = +\infty.$$

Let $J_* := \frac{\alpha_*}{\gamma(\alpha_*)}$. We deduce that

1. *For $J \in [0, J_*)$, δ_0 is the unique invariant probability measure of (1.1).*
2. *For $J \in (J_*, \infty)$, (1.1) has three invariant probability measures: $\{\delta_0,\ \nu^\infty_{\alpha_1},\ \nu^\infty_{\alpha_2}\}$. with $\alpha_1 < \alpha_* < \alpha_2$.*
3. *For $J = J_*$, (1.1) has two invariant probability measures: δ_0 and $\nu^\infty_{\alpha_*}$.*

Proof. The graph of the function $\alpha \mapsto \frac{\alpha}{\gamma(\alpha)}$ is plotted Figure 1. Define

$$\forall \alpha \geq 0, \quad V(\alpha) := \alpha \int_0^1 (1+x) e^{\alpha^2 w(x)} \mathrm{d}x. \tag{4.1}$$

Claim It holds that for all $\alpha > 0$

$$\frac{\alpha}{\gamma(\alpha)} = \frac{1}{\alpha} + V(\alpha). \tag{4.2}$$

Proof. First, note that $\frac{\alpha}{\gamma(\alpha)} = \alpha \widehat{H_\alpha}(0)$. In addition, by Lemma 4.3 with $\theta = 0$ and $z = 0$, we find that $\widehat{H_\alpha}(0) = \frac{1}{\alpha^2} + V(\alpha)/\alpha$. □

Define for all $x \in [0, 1)$

$$A(x) := \frac{-4w(x)}{x^3} - (1+x) = \frac{1}{3} + 4x^2 \sum_{k \geq 0} \frac{x^k}{k+5}.$$

Claim It holds that

$$V'(\alpha) = \int_0^1 A(x) e^{\alpha^2 w(x)} \mathrm{d}x.$$

In particular V is strictly increasing on $\mathbb{R}_+$.
Proof. We have

$$V'(\alpha) = \int_0^1 (1+x) e^{\alpha^2 w(x)} \mathrm{d}x + 2\alpha^2 \int_0^1 (1+x) w(x) e^{\alpha^2 w(x)} \mathrm{d}x.$$

Let

$$\forall x \in (0,1), \quad \theta(x) := \frac{(1+x)w(x)}{w'(x)}.$$

We have $\frac{w(x)}{w'(x)} = -\frac{(1-x)w(x)}{x^2}$ and so $\theta(x) = -\frac{1-x^2}{x^2} w(x)$. In particular, θ can be extended to a $\mathcal{C}^1([0,1])$ function with $\theta(0) = \theta(1) = 0$. Integrating par parts, we find that

$$\begin{aligned} 2\alpha^2 \int_0^1 (1+x) w(x) e^{\alpha^2 w(x)} \mathrm{d}x &= 2\alpha^2 \int_0^1 \theta(x) w'(x) e^{\alpha^2 w(x)} \mathrm{d}x \\ &= -2 \int_0^1 \theta'(x) e^{\alpha^2 w(x)} \mathrm{d}x. \end{aligned}$$

Moreover, we have $\theta'(x) = \frac{2}{x^3} w(x) + (1+x)$ and so $(1+x) - 2\theta'(x) = A(x)$. □

For all $\alpha \geq 1$, we have

$$V'(\alpha) \geq \frac{1}{3} \int_0^1 e^{\alpha^2 w(x)} dx \geq \frac{1}{6\alpha} \alpha \int_0^1 (1+x) e^{\alpha^2 w(x)} dx = \frac{1}{6\alpha} V(\alpha).$$

Consequently, we have $\forall \alpha \geq 1,\ V(\alpha) \geq V(1) \alpha^{1/6}$. Using (4.2), we deduce that

$$\lim_{\alpha \downarrow 0} \frac{\alpha}{\gamma(\alpha)} = +\infty, \quad \text{and} \quad \lim_{\alpha \to \infty} \frac{\alpha}{\gamma(\alpha)} = +\infty.$$

It remains to study the variations of $\alpha \mapsto \frac{\alpha}{\gamma(\alpha)}$. Using (4.2), we have

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} \frac{\alpha}{\gamma(\alpha)} = \frac{\alpha^2 V'(\alpha) - 1}{\alpha^2} = \frac{W(\alpha^2) - 1}{\alpha^2}, \quad \text{with} \quad W(\alpha) := \alpha \int_0^1 A(x) e^{\alpha w(x)} \mathrm{d}x.$$

Claim The function W is increasing on $\mathbb{R}_+$.
Proof. Let $D(x) := \frac{A(x)w(x)}{w'(x)}$. We have

$$W'(\alpha) = \int_0^1 A(x)e^{\alpha w(x)}\mathrm{d}x + \int_0^1 D(x)\alpha w'(x)e^{\alpha w(x)}\mathrm{d}x$$
$$= \int_0^1 [A(x) - D'(x)]e^{\alpha w(x)}\mathrm{d}x.$$

To conclude it suffices to show that for all $x \in [0,1),\ A(x) - D'(x) \geq 0$, which follows from the explicit formula satisfied by A and D. □

Finally, we have $\lim_{\alpha\to\infty} W(\alpha) = +\infty$. This follows from

$$W(\alpha^2) = \alpha^2 V'(\alpha) \geq \alpha^2 \frac{1}{6\alpha} V(1)\alpha^{1/6}.$$

Putting altogether, we deduce the result. □

4.2 Conjecture on their stability

Let $J \in (J_*, \infty)$. By Proposition 4.4, (1.1) has exactly three invariant probability measures: $\{\delta_0, \nu^\infty_{\alpha_1}, \nu^\infty_{\alpha_2}\}$ with $\alpha_1 < \alpha_* < \alpha_2$. It is known that δ_0 is attractive, see [20]. The question of the stability of $\nu^\infty_{\alpha_1}$ and $\nu^\infty_{\alpha_2}$ is more delicate. In view of Theorem 2.8, the stability is determined by the location of the zeros of

$$F(\alpha, z) := \widehat{H}_\alpha(z) - \widehat{\Psi}_\alpha(z).$$

The explicit expression of $F(\alpha, z)$ is given in Proposition 4.2 above. Recall the definition of λ'_α (2.7):

$$-\lambda'_\alpha = \sup\{\Re(z);\quad F(\alpha, z) = 0\}.$$

Conjecture 4.5. *We conjecture that $\lambda'_{\alpha_1} > 0$ and that $\lambda'_{\alpha_2} < 0$.*

In view of Theorem 2.8, this suggests that $\nu^\infty_{\alpha_1}$ is unstable and that $\nu^\infty_{\alpha_2}$ is stable. This conjecture is motivated by numerical investigations, see Figure 2, and by the following analysis for α close to α_*. First we note that for all $\alpha > 0$, it holds that:

$$F(\alpha, 0) = \frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\alpha}{\gamma(\alpha)}.$$

In particular, for $\alpha = \alpha_*$, we have

$$F(\alpha_*, 0) = 0.$$

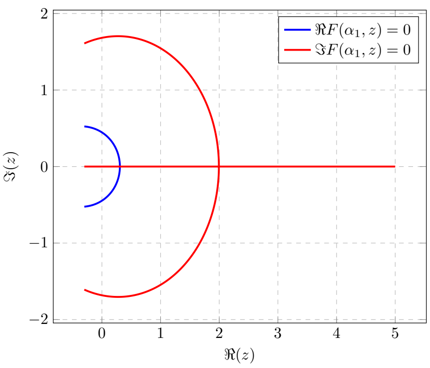
The function $(\alpha, z) \mapsto F(\alpha, z)$ is C^1 in the neighborhood of $(\alpha_*, 0)$. In addition, we find that

$$\partial_z F(\alpha_*, 0) > 0 \quad \text{and} \quad \partial_\alpha F(\alpha_*, 0) > 0.$$

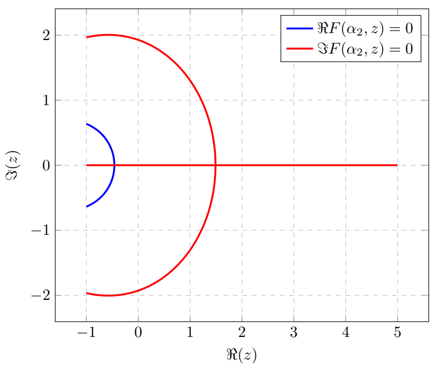
Therefore, the implicit function theorem applies, and gives the existence of a function $\alpha \mapsto z(\alpha)$ in the neighborhood of α_* such that $F(\alpha, z(\alpha)) = 0$. In addition, we have

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} z(\alpha_*) = -\frac{\partial_\alpha F(\alpha_*, 0)}{\partial_z F(\alpha_*, 0)} < 0.$$

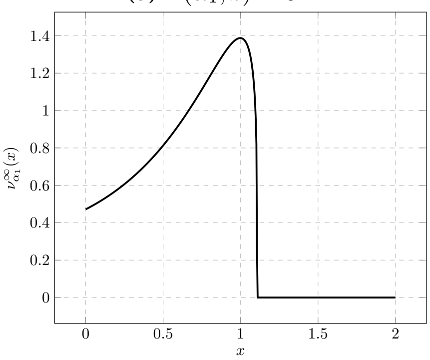
This implies that for $\alpha < \alpha_*$, α sufficiently close to α_*, it holds that $z(\alpha) > 0$ and so $\lambda'_\alpha < 0$. Note however that when $\alpha < \alpha_*$, we have $z(\alpha) < 0$ but this local analysis is not sufficient to conclude that $\lambda'_\alpha > 0$. Indeed, there might be other solutions to the equation $F(\alpha, z) = 0$ with $\Re(z) \geq 0$ and $|z|$ far from zero.



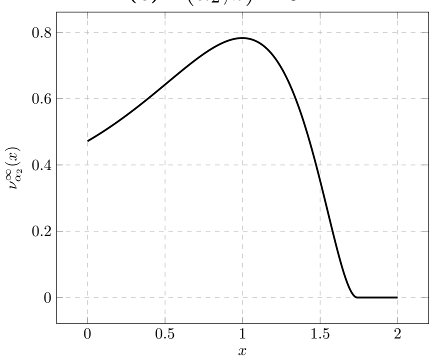
(a) $F(\alpha_1, z) = 0$



(b) $F(\alpha_2, z) = 0$



(c) $\nu_{\alpha_1}^{\infty}(x)$



(d) $\nu_{\alpha_2}^{\infty}(x)$

Figure 2: Let $f(x) = x^2$ and $b(x) = -x$. For $J = 2.12$, the invariant probability measures of (1.1) are $\{\delta_0, \nu_{\alpha_1}^{\infty}, \nu_{\alpha_2}^{\infty}\}$ with $\alpha_1 \approx 1.108$ and $\alpha_2 \approx 1.7383$. The shape of the non-trivial invariant probability measures $\nu_{\alpha_1}^{\infty}$ and $\nu_{\alpha_2}^{\infty}$ are reported in Figures 2c and 2d respectively. Figure 2a, we plot the curves $\Re F(\alpha_1, z) = 0$ (in blue) and $\Im F(\alpha_1, z)$ (in red). The two curves intersect at a zero of $F(\alpha_1, \cdot)$. We find numerically that $F(\alpha_1, 0.3065) \approx 0$. This suggests that $\nu_{\alpha_1}^{\infty}$ is unstable. For $\alpha = \alpha_2$, we find in Figure 2b that the zeros of $z \mapsto F(\alpha_2, z)$ have negative real part, suggesting that $\nu_{\alpha_2}^{\infty}$ is stable.

References

[1] María José Cáceres and Benoît Perthame, *Beyond blow-up in excitatory integrate and fire neuronal networks: refractory period and spontaneous activity*, J. Theoret. Biol. **350** (2014), 81–89, doi. MR-3190511

[2] José Antonio Carrillo, Benoît Perthame, Delphine Salort, and Didier Smets, *Qualitative properties of solutions for the noisy integrate and fire model in computational neuroscience*, Nonlinearity **28** (2015), no. 9, 3365–3388, doi. MR-3403402

[3] Quentin Cormier, *On the stability of the invariant probability measures of Mckean-Vlasov equations*, to appear in Ann. Inst. Henri Poincaré Probab. Stat. (2025), https://arxiv.org/abs/2201.11612.

[4] Quentin Cormier, Etienne Tanré, and Romain Veltz, *Long time behavior of a mean-field model of interacting neurons*, Stochastic Process. Appl. **130** (2020), no. 5, 2553–2595, doi. MR-4080722

[5] ______, *Hopf bifurcation in a mean-field model of spiking neurons*, Electron. J. Probab. **26** (2021), Paper No. 121, 40, doi. MR-4316639

[6] Mark H. A. Davis, *Piecewise-deterministic Markov processes: a general class of non-diffusion stochastic models*, J. Roy. Statist. Soc. Ser. B **46** (1984), no. 3, 353–388, https://www.jstor.org/stable/2345677. MR-790622

[7] Anna De Masi, Antonio Galves, Eva Löcherbach, and Errico Presutti, *Hydrodynamic limit for interacting neurons*, J. Stat. Phys. **158** (2015), no. 4, 866–902, doi. MR-3311484

[8] Audric Drogoul and Romain Veltz, *Hopf bifurcation in a nonlocal nonlinear transport equation stemming from stochastic neural dynamics*, Chaos **27** (2017), no. 2, 021101, 6, doi. MR-3612404

[9] ______, *Exponential stability of the stationary distribution of a mean field of spiking neural network*, J. Differential Equations **270** (2021), 809–842, doi. MR-4152223

[10] Aline Duarte and Guilherme Ost, *A model for neural activity in the absence of external stimuli*, Markov Process. Related Fields **22** (2016), no. 1, 37–52. MR-3523978

[11] Grégory Dumont and Pierre Gabriel, *The mean-field equation of a leaky integrate-and-fire neural network: measure solutions and steady states*, Nonlinearity **33** (2020), no. 12, 6381–6420, doi.

[12] Nicolas Fournier and Eva Löcherbach, *On a toy model of interacting neurons*, Ann. Inst. Henri Poincaré Probab. Stat. **52** (2016), no. 4, 1844–1876, doi. MR-3573298

[13] Antonio Galves and Eva Löcherbach, *Infinite systems of interacting chains with memory of variable length—a stochastic model for biological neural nets*, J. Stat. Phys. **151** (2013), no. 5, 896–921, doi. MR-3055382

[14] Wulfram Gerstner, Werner M. Kistler, Richard Naud, and Liam Paninski, *Neuronal dynamics: From single neurons to networks and models of cognition*, Cambridge University Press, 2014.

[15] Gustaf Gripenberg, Stig-Olof Londen, and Olof Staffans, *Volterra integral and functional equations*, Encyclopedia of Mathematics and its Applications, vol. 34, Cambridge University Press, Cambridge, 1990, doi. MR-1050319

[16] Pierre Hodara, Nathalie Krell, and Eva Löcherbach, *Non-parametric estimation of the spiking rate in systems of interacting neurons*, Stat. Inference Stoch. Process. **21** (2018), no. 1, 81–111, doi. MR-3769833

[17] Pierre Hodara and Ioannis Papageorgiou, *Poincaré-type inequalities for compact degenerate pure jump markov processes*, Mathematics **7** (2019), no. 6, 1–18, doi.

[18] Eva Löcherbach and Pierre Monmarché, *Metastability for systems of interacting neurons*, Ann. Inst. Henri Poincaré Probab. Stat. **58** (2022), no. 1, 343–378, doi. MR-4374679

[19] Stéphane Mischler and Qilong Weng, *Relaxation in time elapsed neuron network models in the weak connectivity regime*, Acta Appl. Math. **157** (2018), 45–74, doi. MR-3850019

[20] Philippe Robert and Jonathan Touboul, *On the dynamics of random neuronal networks*, J. Stat. Phys. **165** (2016), no. 3, 545–584, doi. MR-3562424

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