Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2025, Vol. 61, No. 4, 2405–2429
https://doi.org/10.1214/24-AIHP1504
© Association des Publications de l'Institut Henri Poincaré, 2025

On the stability of the invariant probability measures of McKean–Vlasov equations

Quentin Cormier[a]

Inria, CMAP, CNRS, École polytechnique, Institut Polytechnique de Paris, 91200 Palaiseau, France, [a]*quentin.cormier@inria.fr*

Received 5 July 2023; revised 12 April 2024; accepted 26 April 2024

Abstract. We study the long-time behavior of some McKean–Vlasov stochastic differential equations used to model the evolution of large populations of interacting agents. We give conditions ensuring the local stability of an invariant probability measure. Lions derivatives are used in a novel way to obtain our stability criteria. We obtain results for non-local McKean–Vlasov equations on $\mathbb{R}^d$ and for McKean–Vlasov equations on the torus where the interaction kernel is given by a convolution. On $\mathbb{R}^d$, we prove that the location of the roots of an analytic function determines the stability. On the torus, our stability criterion involves the Fourier coefficients of the interaction kernel. In both cases, we prove the convergence in the Wasserstein metric W_1 with an exponential rate of convergence.

Résumé. On étudie le comportement en temps long d'équations non-linéaires au sens de McKean–Vlasov. Ces équations sont utilisées pour modéliser les comportements de grandes populations d'agents en interaction. On donne des critères garantissant la stabilité locale d'une mesure de probabilité invariante. Pour ce faire, on utilise de manière novatrice les dérivées de Lions. On obtient des résultats pour des équations de McKean–Vlasov non locales sur $\mathbb{R}^d$, ainsi que pour des équations de McKean–Vlasov sur le tore pour lesquelles le noyau d'interaction est donné par une convolution. Sur $\mathbb{R}^d$, on montre que la stabilité est déterminée par l'emplacement des zéros d'une fonction holomorphe. Sur le tore, notre critère de stabilité fait intervenir les coefficients de Fourier du noyau d'interaction. Dans les deux cas, on montre la convergence vers la mesure de probabilité invariante à vitesse exponentielle pour la métrique de Wasserstein W_1.

MSC2020 subject classifications: Primary 60H10; secondary 60K35; 45D05; 37A30

Keywords: McKean–Vlasov SDE; Long-time behavior; Mean-field interaction; Lions derivative

1. Introduction

We are interested in the long-time behavior of the solutions of a class of McKean–Vlasov stochastic differential equations (SDE) of the form:

$$\begin{aligned} \mathrm{d}X_t^\nu &= \mathcal{V}\big(X_t^\nu, \mu_t\big)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t, \\ \mu_t &= \mathrm{Law}\big(X_t^\nu\big), \quad \mu_0 = \nu. \end{aligned} \tag{1}$$

In this equation, $(B_t)_{t\geq 0}$ is a standard $\mathbb{R}^d$-valued Brownian motion, σ is a deterministic matrix, and ν is the law of the initial condition X_0^ν, assumed to be independent of $(B_t)_{t\geq 0}$. McKean–Vlasov equations appear naturally as the limit $N \to \infty$ of the following particle system $(X_t^{i,N})_{t\geq 0}$, solution of

$$\mathrm{d}X_t^{i,N} = \mathcal{V}\big(X_t^{i,N}, \mu_t^N\big)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t^{i,N}, \quad 1 \leq i \leq N, \tag{2}$$

where μ_t^N is the empirical measure $\mu_t^N = \frac{1}{N}\sum_{j=1}^N \delta_{X_t^{j,N}}$ and $(B_t^{i,N})_{t\geq 0}$ are N independent standard Brownian motions. We refer to [45] for an introduction to this topic.

Such particle systems and their mean-field counterparts are used in a wide range of applications such as plasma physics [22,32], fluid mechanics [25], astrophysics (particles are stars or galaxies [51]), bio-sciences (to understand the collective behavior of animals [8]), neuroscience (to model assemblies of neurons, such as integrate and fire neurons [20, 26] or FitzHugh–Nagumo neurons [39]), opinion dynamics [18] and economics [10].

In these applications, one important question concerns the long-time behavior of the solutions. As such, the ergodic properties of McKean–Vlasov equations (1) have been studied in many different contexts and approaches.

Two families of assumptions are known to ensure that (1) admits a unique, globally attractive invariant probability measure. The first type of assumption deals with kernels given by $\mathcal{V}(x,\mu) = -\nabla V(x) - \nabla W * \mu(x)$, where V, W have suitable convexity properties. The first results in this direction were obtained in [3,4] in dimension one. In larger dimensions, [41,50] proved the convergence uniformly in time of a suitable particle system towards the mean-field equation. As such, they obtained the ergodicity of the McKean–Vlasov equation from the ergodicity of the particle system. These uniformly in time propagation of chaos arguments have been used wisely; see for instance [19,31] for recent results in this direction. These results have also been obtained by using functional inequalities [5,12]: the idea is to define a measure-valued functional (known as the entropy or free energy), which only decreases along the trajectories of the solution of (1).

The second kind of assumption involves weak enough interactions. When the dependence of $\mathcal{V}$ with respect to the measure is sufficiently weak, one expects global stability because this situation can be seen as a perturbation of the case without interactions. As such, it is possible to extend techniques from ergodic Markov processes to the case of weak interactions. This includes, for instance, coupling techniques [1,9,23,24,27] or Picard iterations in suitable spaces [16].

It is also well-known that, in general, such global stability results cannot hold because (1) may have multiple invariant probability measures and periodic solutions [34,43,49]. These examples motivate the current question of the paper, namely the study of the local stability of a given invariant probability measure of (1). That is, being given ν_∞ an invariant probability measure of (1), we address the following question:

Is there exist an open neighborhood of ν_∞ such that for all initial conditions ν within this neighborhood, the law of X^ν_t converges to ν_∞, as t goes to infinity? If so, for which metric does the convergence hold, and what is the rate of convergence?

Such local stability results can be obtained via partial differential equation (PDE) techniques, using that the marginals of the non-linear process solve a non-linear PDE (the Fokker–Planck equation). The strategy is to linearize the non-linear PDE around ν_∞, to study the existence of a spectral gap for the linear equation in appropriate Banach spaces, and to use perturbation techniques to obtain the convergence for the non-linear PDE. We refer to [33,40] for an overview of these techniques. When the non-linear PDE admits a gradient flow structure, it is also possible to study the local stability of an invariant probability measure using functional inequalities; see [11,47,48]. In [47], the author study the local stability of an invariant probability measure in weighted L^2 norm. The result is obtained assuming a spectral condition related to the positivity of the Hessian of the free energy functional, evaluated at the invariant distribution.

Our approach differs from these two methods on several points. We do not rely on the non-linear Fokker–Planck PDE nor need a gradient flow structure. Instead, we use directly the stochastic representation (1). Our strategy is to differentiate the interaction kernel with respect to the initial probability measure, in the neighborhood of ν_∞. There are several notions of derivation with respect to probability measures (see [10]): we use here the Lions derivatives. We denote by $\mathcal{P}_2(\mathbb{R}^d)$ the set of probability measures on $\mathbb{R}^d$ having a second moment. For all $x \in \mathbb{R}^d$ and $t \geq 0$, we consider the function

$$\mathcal{P}_2(\mathbb{R}^d) \ni \nu \mapsto \mathcal{V}\big(x, \mathrm{Law}\big(X^\nu_t\big)\big) =: v^x_t(\nu) \in \mathbb{R}^d,$$

where X^ν_t is the solution of (1) starting with law ν at time 0. We prove that under suitable assumptions, this function is Lions differentiable at ν_∞, meaning that for all $\nu \in \mathcal{P}_2(\mathbb{R}^d)$, we have

$$\mathcal{V}\big(x, \mathrm{Law}\big(X^\nu_t\big)\big) = \mathcal{V}(x, \nu_\infty) + \mathbb{E}\partial_\nu v^x_t(\nu_\infty)(Z_0) \cdot (Z - Z_0) + o\big(\big(\mathbb{E}|Z - Z_0|^2\big)^{1/2}\big).$$

In this equation, Z, Z_0 are any random variables defined on the same probability space, with laws equal to ν and ν_∞. We write $\mathbb{E}(Z - Z_0|Z_0) = h(Z_0)$, where h is a deterministic function from $\mathbb{R}^d$ to $\mathbb{R}^d$. As such, the function h encodes the correlations between the initial conditions Z and Z_0. It follows from the Cauchy–Schwarz inequality that $\mathbb{E}|h(Z_0)|^2 \leq \mathbb{E}|Z - Z_0|^2 < \infty$. Therefore, $h \in L^2(\nu_\infty)$. We define the linear operator $\Omega_t : L^2(\nu_\infty) \to L^2(\nu_\infty)$ by

$$\Omega_t(h) := x \mapsto \mathbb{E}\partial_\nu v^x_t(\nu_\infty)(Z_0) \cdot h(Z_0).$$

The fact that $\Omega_t(h) \in L^2(\nu_\infty)$ for all $h \in L^2(\nu_\infty)$ is not granted apriori and will follow from our assumptions on the function $\mathcal{V}$. So we have (recall that $\nu = \mathrm{Law}(Z)$ and $\mathbb{E}(Z - Z_0|Z_0) = h(Z_0)$)

$$\mathcal{V}\big(x, \mathrm{Law}\big(X^\nu_t\big)\big) = \mathcal{V}(x, \nu_\infty) + \Omega_t(h)(x) + o\big(\big(\mathbb{E}|Z - Z_0|^2\big)^{1/2}\big).$$

Our spectral conditions under which we prove that ν_∞ is locally stable can be stated in terms of the decay of the function $t \mapsto \Omega_t$, as t goes to infinity. We show that the integrability of this function on $\mathbb{R}_+$ implies the stability of ν_∞. In addition, the decay of $t \mapsto \Omega_t$ as t goes to infinity gives precisely the rate of convergence of $\mathrm{Law}(X_t^\nu)$ towards ν_∞, in Wasserstein metrics. Crucial to our analysis, we provide an explicit integral equation to compute this function Ω_t. To do so, we consider the linear process $(Y_t^\nu)_{t\geq 0}$ associated with (1) and ν_∞, defined as the solution of

$$\mathrm{d}Y_t^\nu = \mathcal{V}\big(Y_t^\nu, \nu_\infty\big)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t,$$

starting from $\mathrm{Law}(Y_0^\nu) = \nu$. We define similarly for $x \in \mathbb{R}^d$ and $t \geq 0$ the function

$$\mathcal{P}_2\big(\mathbb{R}^d\big) \ni \nu \mapsto u_t^x(\nu) := \mathcal{V}\big(x, \mathrm{Law}\big(Y_t^\nu\big)\big).$$

Under non-restrictive assumptions, u_t^x is Lions differentiable at ν_∞, and we can define

$$\forall h \in L^2(\nu_\infty), \quad \Theta_t(h) := x \mapsto \mathbb{E}\partial_\nu u_t^x(\nu_\infty)(Z_0)\cdot h(Z_0).$$

We prove the following key relation between Θ_t and Ω_t

$$\forall t \geq 0, \quad \Omega_t(h) = \Theta_t(h) + \int_0^t \Theta_{t-s}\big(\Omega_s(h)\big)\,\mathrm{d}s.$$

That is, Ω is a solution of a Volterra integral equation whose kernel is given by Θ: in the language of integral equations, Ω is the resolvent of Θ. This relation is helpful because it is easier to get estimates on u_t^x, which involves a linear Markov process, rather than getting estimates on v_t^x, which involves the solution of the McKean–Vlasov equation (1). In particular, this relation allows to deduce the decay properties of Ω from properties of Θ, using Laplace transform. We obtain our stability results for the Wasserstein W_1 metric.

The contributions of this work are the following. First, in Section 2, we consider dynamics of the form $\mathcal{V}(x,\mu) = b(x) + \int_{\mathbb{R}^d} F(x,y)\mu(\mathrm{d}y)$, for some smooth functions $b : \mathbb{R}^d \to \mathbb{R}^d$ and $F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$. The function b is assumed to be confining. Our main result, Theorem 2.5, states that the stability of an invariant probability measure is determined by the location of the roots of an explicit analytic function associated with the dynamics. Stability holds when all the roots lie on the left half-plane, and we prove convergence in Wassertein metric W_1 with an exponential rate. Our result shows that the stability is completely determined by a discrete set, this set being given by the zeros of an analytic function associated to the underlying Markov process. Note that we do not require any structural assumption on b and F: in particular, we do not require any convexity assumption on the coefficients. Our stability criterion is analogous to the Jacobian stability criterion for ODE, for which the location of the zeros of the characteristic polynomial determines the stability.

Second, in Section 3, we consider a McKean–Vlasov equation on the torus $\mathbb{T}^d := (\mathbb{R}/(2\pi\mathbb{Z}))^d$, with an interaction kernel given by a convolution: $\mathcal{V}(x,\mu) = -\int_{\mathbb{T}^d} \nabla W(x-y)\mu(\mathrm{d}y)$, where W is a smooth function from $\mathbb{T}^d$ to $\mathbb{R}$. We assume that $\sigma = \sqrt{2\beta^{-1}} I_d$ for some $\beta > 0$, where I_d is the identity matrix. This setting covers many interesting models; see [11]. We study the stability of the uniform probability measure $U(\mathrm{d}x) := \frac{\mathrm{d}x}{(2\pi)^d}$. Our second main result, Theorem 3.2, states that when $\inf_{n\in\mathbb{Z}^d\setminus\{0\}} |n|^2(\beta + \tilde{W}(n)) > 0$, $\tilde{W}(n)$ being the n-th Fourier coefficient of W, then U is locally stable for the W_1 metric. Our result complements the results of [11], for which static bifurcations are studied: in particular, we exhibit the same critical parameter. In both parts, we use the strategy described above, using Lions derivatives and probabilistic tools. The criteria we obtain are optimal: violations of the criteria occur strictly at bifurcation points.

The strategy presented in this work also applies to mean-field models of noisy integrate-and-fire neurons: in [15], we study the stability of the stationary solutions of such a mean-field model of noisy neurons by applying the methodology developped here. In addition, periodic solutions via Hopf bifurcations are studied in [17]. For the sake of clarity, we restrict here ourselves to a diffusive setting.

Finally, we mention an important open problem concerning the long-time behavior of the particle system (2). On the one hand, general conditions are known to ensure that the particle system is ergodic. On the other hand, numerical studies show that this particle system can have metastable behavior in the sense that the convergence of the empirical measure μ_t^N towards its invariant state can be very slow when N is large. The locally stable invariant probability measures of the non-linear equation (1) are good candidates to be metastable states of the particle system (2). Characterizing those metastable states in quantitative terms is a challenging mathematical question. Recent partial results have been obtained in this direction [2,7,14,21,37], and we hope to progress on this question in future works.

2. McKean–Vlasov equations on $\mathbb{R}^d$

2.1. *Main result*

Let $\mathcal{P}_1(\mathbb{R}^d)$ be the space of probability measures on $\mathbb{R}^d$ with a finite first moment. We consider the following McKean–Vlasov equation on $\mathbb{R}^d$:

$$\mathrm{d}X_t^\nu = b(X_t^\nu)\,\mathrm{d}t + \int_{\mathbb{R}^d} F(X_t^\nu, y)\mu_t(\mathrm{d}y)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t \quad \text{with } \mu_t = \mathrm{Law}(X_t^\nu). \tag{3}$$

The initial condition X_0^ν has law $\nu \in \mathcal{P}_1(\mathbb{R}^d)$. Here, $(B_t)_{t\geq 0}$ is a d-dimensional standard Brownian motion, $\sigma \in M_d(\mathbb{R})$ is a constant $d \times d$ matrix with $\det\sigma > 0$.

Assumption 2.1. The functions $b : \mathbb{R}^d \to \mathbb{R}^d$ and $F : \mathbb{R}^{2d} \to \mathbb{R}^d$ are C^2, b is globally Lipschitz, and the derivatives of F are bounded (b and F are not assumed to be bounded themselves):

$$\forall i, j \in \{i, \dots, d\}^2, \quad \|\partial_{x_i} F\|_\infty + \|\partial^2_{x_i,x_j} F\|_\infty < \infty.$$

This ensures in particular that (3) has a unique path-wise solution. Let $\nu_\infty \in \mathcal{P}_1(\mathbb{R}^d)$ be an invariant probability measure of (3), that is:

$$\forall t \geq 0, \quad \mathrm{Law}(X_t^{\nu_\infty}) = \nu_\infty.$$

Denote by $\alpha(x)$ the interaction term under ν_∞:

$$\forall x \in \mathbb{R}^d, \quad \alpha(x) := \int_{\mathbb{R}^d} F(x, y)\nu_\infty(\mathrm{d}y). \tag{4}$$

Each invariant probability measure of (3) is characterized by its associated function α, and we sometimes denote by ν_∞^α such invariant probability measure to emphasize the dependence on α. We assume:

Assumption 2.2. There exists $\beta > 0$ and $R \geq 0$ such that

$$\forall x, x' \in \mathbb{R}^d, \quad |x - x'| \geq R \quad \Longrightarrow \quad (x - x') \cdot [(b + \alpha)(x) - (b + \alpha)(x')] \leq -\beta|x - x'|^2.$$

Remark 2.3. In particular, this is satisfied provided that F is bounded (or independent of x) and there exists $\beta > 0$ and $R \geq 0$ such that:

$$|x - x'| \geq R \quad \Longrightarrow \quad (x - x') \cdot (b(x) - b(x')) \leq -\beta|x - x'|^2.$$

Let $(Y_t^\alpha)_{t\geq 0}$ the solution of the linear SDE

$$\mathrm{d}Y_t^\alpha = b(Y_t^\alpha)\,\mathrm{d}t + \alpha(Y_t^\alpha)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t. \tag{5}$$

Note that $\nu_\infty = \nu_\infty^\alpha$ is also an invariant probability measure of this linear SDE. In addition, a result of Eberle [23] (see (9) below) ensures that ν_∞ is the unique invariant probability measure of (5).

Consider $\mathcal{H} := L^2(\nu_\infty)$ the Hilbert space of measurable functions $h : \mathbb{R}^d \to \mathbb{R}^d$ satisfying:

$$\|h\|^2_{\mathcal{H}} := \int |h(y)|^2 \nu_\infty(\mathrm{d}y) < \infty.$$

We denote by $\mathcal{L}(\mathcal{H})$ the space of bounded linear operators from $\mathcal{H}$ to itself. Key to our analysis is the following family of bounded linear operators $\Theta_t \in \mathcal{L}(\mathcal{H})$, $t \geq 0$:

$$\forall h \in \mathcal{H}, \quad \Theta_t(h)(x) := \int_{\mathbb{R}^d} \nabla_y \mathbb{E}_y F(x, Y_t^\alpha) \cdot h(y)\nu_\infty^\alpha(\mathrm{d}y). \tag{6}$$

The notation $\mathbb{E}_y F(x, Y_t^\alpha)$ means that the initial condition of (Y_t^α) is set to be $y \in \mathbb{R}^d$ (that is $Y_0^\alpha = y$). In addition, $\nabla_y \mathbb{E}_y F(x, Y_t^\alpha)$ is the Jacobian matrix of $y \mapsto \mathbb{E}_y F(x, Y_t^\alpha)$. The result of Eberle [23] (see (10) below) implies that there exists $\kappa_* > 0$ and $C > 0$ such that

$$\forall h \in \mathcal{H}, \quad \|\Theta_t(h)\|_\infty \leq Ce^{-\kappa_* t}\|h\|_{\mathcal{H}}. \tag{7}$$

Denote by $I \in \mathcal{L}(\mathcal{H})$ the identity operator. Let $D = \{z \in \mathbb{C},\ \Re(z) > -\kappa_*\}$. For all $z \in D$, consider the Laplace transform of Θ_t:

$$\hat{\Theta}(z) = \int_0^\infty e^{-zt}\Theta_t \, \mathrm{d}t.$$

We recall some properties of this operator-valued function.

Proposition 2.4. *We have*:

1. *For all $z \in D$, $\hat{\Theta}(z) \in \mathcal{L}(\mathcal{H})$ is an Hilbert–Schmidt operator. In particular, $\hat{\Theta}(z)$ is a compact operator.*
2. *$D \ni z \mapsto \hat{\Theta}(z)$ is an analytic operator-valued function, and $(I - \hat{\Theta}(z))^{-1}$ exists for all $z \in D \setminus S$, where S is a discrete subset of D. In addition, $(I - \hat{\Theta}(z))^{-1}$ is meromorphic in D, holomorphic in $D \setminus S$.*
3. *The function $D \ni z \mapsto \det(I - \hat{\Theta}(z)) \in \mathbb{C}$ is analytic, where $\det$ is the regularized Fredholm determinant. In addition, $I - \hat{\Theta}(z)$ is invertible if and only if $\det(I - \hat{\Theta}(z)) \neq 0$.*

Proof. By Fubini, we have

$$\hat{\Theta}(z)(h) = x \mapsto \int_{\mathbb{R}^d} K_z(x, y) h(y) \nu_\infty(\mathrm{d}y),$$

where $K_z(x, y) := \int_0^\infty e^{-zt}\nabla_y \mathbb{E}_y F(x, Y_t^\alpha) \, \mathrm{d}t$. Using (10), it holds that $\sup_{x,y\in\mathbb{R}^d} |K_z(x, y)| < \infty$ for all $z \in D$. Therefore, $K_z(\cdot, \cdot) \in L^2(\mathbb{R}^d \times \mathbb{R}^d, \nu_\infty \otimes \nu_\infty)$ and so Theorem VI.23 in [42] applies and gives the first point. For $\Re(z)$ large enough, it holds that $\|\hat{\Theta}(z)\|_{\mathcal{H}} < 1$. Therefore, $I - \hat{\Theta}(z)$ is invertible provided that $\Re(z)$ is large enough, the inverse is given by its Neumann series. So the second point follows from the analytic Fredholm theorem, see Theorem VI.14 in [42]. Finally, the third point is a fundamental result of the theory of Fredholm determinants for Hilbert–Schmidt operators, see for instance [44]. □

Our main result is

Theorem 2.5. *Consider ν_∞ an invariant probability measure of (3) and let α be given by (4). Assume that Assumptions 2.1 and 2.2 hold. Define the "abscissa" of the rightmost zeros of* $\det(I - \hat{\Theta}(z))$:

$$-\lambda' := \sup\bigl\{\Re(z) \mid z \in D,\ \det\bigl(I - \hat{\Theta}(z)\bigr) = 0\bigr\}. \tag{8}$$

Assume that $\lambda' > 0$. Then ν_∞ is locally stable: there exists $C, \epsilon > 0$ and $\lambda \in (0, \lambda')$ such that for all $\nu \in \mathcal{P}_1(\mathbb{R}^d)$ with $W_1(\nu, \nu_\infty) < \epsilon$, it holds that

$$\forall t \geq 0, \quad W_1\bigl(\mathrm{Law}\bigl(X_t^\nu\bigr), \nu_\infty\bigr) \leq C W_1(\nu, \nu_\infty) e^{-\lambda t}.$$

2.2. *Remarks and examples*

We now give explanations on Theorem 2.5, in particular on the spectral assumption involving (8). From now on, the constants may vary from one line to the other.

Gradients bounds

We denote by (Y_t^{α,δ_x}) the solution of (5) with initial condition $Y_0^{\alpha,\delta_x} = x$. Under assumptions 2.2, Theorem 1 in [23] applies: there exists $\kappa_* > 0$ and $C_* > 1$ such that for all $x, y \in \mathbb{R}^d$ and all $t \geq 0$,

$$W_1\bigl(\mathrm{Law}\bigl(Y_t^{\alpha,\delta_x}\bigr), \mathrm{Law}\bigl(Y_t^{\alpha,\delta_y}\bigr)\bigr) \leq C_* e^{-\kappa_* t}|x - y|. \tag{9}$$

We deduce from this inequality the following gradient bound. For all $f \in C^1(\mathbb{R}^d)$:

$$\forall y \in \mathbb{R}^d, \quad \bigl|\nabla_y \mathbb{E}_y f\bigl(Y_t^\alpha\bigr)\bigr| \leq C_* \|\nabla f\|_\infty e^{-\kappa_* t}. \tag{10}$$

In particular, by choosing $f = F(x, \cdot)$, we obtain the estimate (7). Note that it is possible to get gradient bounds similar to (10) under less restrictive assumptions on b and σ; see [46].

On the spectral condition
We consider the following bounded linear operator $\Omega_t \in \mathcal{L}(\mathcal{H})$ by taking the Neumann series:

$$\forall h \in \mathcal{H}, \quad \Omega_t(h) := \sum_{i \geq 1} \Theta_t^{\otimes i}(h), \tag{11}$$

where the linear operators $\Theta_t^{\otimes(i)}$ are defined recursively by

$$\forall t \geq 0, \quad \Theta_t^{\otimes(i+1)}(h) = \int_0^t \Theta_{t-s}\big(\Theta_s^{\otimes i}(h)\big)\,\mathrm{d}s, \quad \text{and} \quad \Theta_t^{\otimes 1}(h) = \Theta_t(h).$$

The series (11) converges uniformly on any compact $[0,T]$ for $T>0$. The operators Ω_t and Θ_t satisfy the following Volterra integral equation:

$$\begin{aligned}\forall h \in \mathcal{H}, \quad \Omega_t(h) &= \Theta_t(h) + \int_0^t \Theta_{t-s}\big(\Omega_s(h)\big)\,\mathrm{d}s \\ &= \Theta_t(h) + \int_0^t \Omega_{t-s}\big(\Theta_s(h)\big)\,\mathrm{d}s.\end{aligned} \tag{12}$$

We denote by $\|\Omega_t\|_{\mathcal{L}(\mathcal{H})}$ the operator norm of Ω_t:

$$\|\Omega_t\|_{\mathcal{L}(\mathcal{H})} := \sup_{\|h\|_{\mathcal{H}} \leq 1} \big\|\Omega_t(h)\big\|_{\mathcal{H}}.$$

Then the spectral condition $\lambda' > 0$ is equivalent to the exponential decay of $t \mapsto \|\Omega_t\|_{\mathcal{L}(\mathcal{H})}$:

Proposition 2.6. *The two following statements are equivalent*

1. $\lambda' > 0$, *where* λ' *is given by* (8).
2. $\exists \lambda > 0$ *such that* $\sup_{t\geq 0} e^{\lambda t}\|\Omega_t\|_{\mathcal{L}(\mathcal{H})} < \infty$.

Proof. We first show that (b) implies (a): by assumption, there exists $\lambda \in (0, \kappa_*)$ such that $z \mapsto \hat{\Omega}(z)$ is an analytic operator-valued function on $\Re(z) > -\lambda$. In view of (12), we have for $\Re(z) > -\lambda$, $\hat{\Omega}(z) = \hat{\Theta}(z) + \hat{\Theta}(z)\cdot\hat{\Omega}(z) = \hat{\Theta}(z) + \hat{\Omega}(z)\cdot\hat{\Theta}(z)$, and so

$$I = \big(I + \hat{\Omega}(z)\big)\big(I - \hat{\Theta}(z)\big) = \big(I - \hat{\Theta}(z)\big)\big(I + \hat{\Omega}(z)\big).$$

Therefore, $I - \hat{\Theta}(z)$ is invertible for all $\Re(z) > -\lambda$, and consequently $\lambda' \geq \lambda > 0$.

We then show that (a) implies (b): this follows from a Paley–Wiener theorem. Let $\lambda \in (0, \lambda')$ and define $K_t := e^{\lambda t}\Theta_t$ and $R_t := e^{\lambda t}\Omega_t$. It holds that $K \in L^1(\mathbb{R}_+; \mathcal{L}(\mathcal{H}))$ (because $\lambda < \kappa_*$) and $I - \hat{K}(z)$ is invertible for all $\Re(z) \geq 0$ (because $\lambda < \lambda'$). Therefore, by [30, Ch. 2, Th. 4.1],[1] it holds that $R \in L^1(\mathbb{R}_+; \mathcal{L}(\mathcal{H}))$. Using the estimate (7) and (12), we find that

$$\begin{aligned}\big\|\Omega_t(h)\big\|_\infty &\leq Ce^{-\kappa_* t}\|h\|_{\mathcal{H}} + C\int_0^t e^{-\kappa_* s}\|\Omega_{t-s}\|_{\mathcal{L}(\mathcal{H})}\|h\|_{\mathcal{H}}\,\mathrm{d}s \\ &\leq C\|h\|_{\mathcal{H}}\bigg(e^{-\kappa_* t} + e^{-\lambda t}\int_0^t e^{\lambda(t-s)}\|\Omega_{t-s}\|_{\mathcal{L}(\mathcal{H})}\,\mathrm{d}s\bigg) \\ &\leq C\|h\|_{\mathcal{H}}\bigg(1 + \int_0^\infty \|R_s\|_{\mathcal{L}(\mathcal{H})}\,\mathrm{d}s\bigg)e^{-\lambda t}.\end{aligned}$$

This shows that $\sup_{t\geq 0} e^{\lambda t}\|\Omega_t\|_{\mathcal{L}(\mathcal{H})} < \infty$. This ends the proof. □

[1]The result in [30] is stated and proved for matrix valued operators. The extension to Hilbert–Schmidt operators is straightforward by the exact same arguments.

The spectral condition is necessary
Let $h \in \mathcal{H}$ be fixed and let Z_0 be a random variable of law ν_∞. For $\epsilon \in \mathbb{R}$, let $\nu_\epsilon := \mathrm{Law}(Z_0 + \epsilon h(Z_0))$. We will see that the operator Ω_t admits the following probabilistic representation (see Remark 2.23 below):

$$\forall t \geq 0, \quad \Omega_t(h)(x) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\mathbb{R}^d} F(x,y)\big(\mathrm{Law}\big(X_t^{\nu_\epsilon}\big) - \nu_\infty\big)(\mathrm{d}y), \quad x \in \mathbb{R}^d, \tag{13}$$

where $(X_t^{\nu_\epsilon})$ is the solution of the McKean–Vlasov equation (3) starting with law ν_ϵ. We have by the Cauchy–Schwarz inequality

$$W_1(\nu_\epsilon, \nu_\infty) \leq \epsilon \mathbb{E}\big|h(Z_0)\big| \leq \epsilon \|h\|_{\mathcal{H}}.$$

Assume that the conclusion of Theorem 2.5 holds. Then there exists $C, \lambda > 0$ such that for all ϵ small enough,

$$W_1\big(\mathrm{Law}\big(X_t^{\nu_\epsilon}\big), \nu_\infty\big) \leq C\epsilon e^{-\lambda t}\|h\|_{\mathcal{H}}.$$

We deduce that:

$$\forall x \in \mathbb{R}^d, \quad \left|\int_{\mathbb{R}^d} F(x,y)\big(\mathrm{Law}\big(X_t^{\nu_\epsilon}\big) - \nu_\infty\big)(\mathrm{d}y)\right| \leq C\epsilon \|\nabla_y F\|_\infty e^{-\lambda t}\|h\|_{\mathcal{H}}.$$

Therefore,

$$\big\|\Omega_t(h)\big\|_\infty \leq C\|\nabla_y F\|_\infty e^{-\lambda t}\|h\|_{\mathcal{H}}.$$

Using Proposition 2.6, we deduce that $\lambda' > 0$, where λ' is given by (8). In other words, the spectral condition $\lambda' > 0$ is necessary to have stability in W_1 norm.

Structural stability with convex coefficients
The classical situation where stability is known to hold globally is the following: assume that there exist functions $V, F : \mathbb{R}^d \to \mathbb{R}$ such that $b(x) = -\nabla V(x)$ and $F(x,y) = -\nabla W(x-y)$. Assume that W is convex and even and that V is strongly convex:

$$\exists \theta > 0, \forall x \in \mathbb{R}^d, \quad W(x) = W(-x), \qquad \nabla^2 W(x) \geq 0, \quad \text{and} \quad \nabla^2 V(x) \geq \theta I_d. \tag{14}$$

Then, by [13, Th. 7], it holds that

$$\forall t \geq 0, \forall \nu, \mu \in \mathcal{P}_2\big(\mathbb{R}^d\big), \quad W_2\big(\mathrm{Law}\big(X_t^\nu\big), \mathrm{Law}\big(X_t^\mu\big)\big) \leq e^{-\theta t/2} W_2(\nu, \mu).$$

Therefore, these structural conditions ensure that (3) has a unique invariant probability measure, which is globally stable in W_2 norm. We show that our spectral condition is satisfied under these conditions.

Lemma 2.7. *Under the structural assumption* (14), *it holds that* $\lambda' > 0$, λ' *given by* (8), *and so Theorem* 2.5 *applies.*

Proof. Let ν_∞ be the unique invariant probability measure of (3). Let $h \in \mathcal{H}$, $\epsilon \in \mathbb{R}$ and let $\nu_\epsilon = \mathcal{L}(Z_0 + \epsilon h(Z_0))$, where $\mathrm{Law}(Z_0) = \nu_\infty$. We have

$$\begin{aligned}\left|\int_{\mathbb{R}^d} \nabla W(x-y)\big(\mathrm{Law}\big(X_t^{\nu_\epsilon}\big) - \nu_\infty\big)(\mathrm{d}y)\right| &\leq \big\|\nabla^2 W\big\|_\infty W_1\big(\mathrm{Law}\big(X_t^{\nu_\epsilon}\big), \nu_\infty\big) \\ &\leq \big\|\nabla^2 W\big\|_\infty W_2\big(\mathrm{Law}\big(X_t^{\nu_\epsilon}\big), \nu_\infty\big) \\ &\leq \big\|\nabla^2 W\big\|_\infty e^{-\theta t/2} W_2(\nu_\epsilon, \nu_\infty) \\ &\leq C e^{-\theta t/2}\sqrt{\mathbb{E}\big|\epsilon h(Z_0)\big|^2} = C|\epsilon| e^{-\theta t/2}\|h\|_{\mathcal{H}}.\end{aligned}$$

Therefore, by (13), we have

$$\forall h \in \mathcal{H}, \quad \big\|\Omega_t(h)\big\|_\infty \leq C e^{-\theta t/2}\|h\|_{\mathcal{H}}.$$

So Proposition 2.6 applies and $\lambda' > 0$. □

Case of weak interactions
One way to check that the spectral condition $\lambda' > 0$ holds, λ' given by (8), is to compute the L^1 norm of Θ_t. Recall that $\|\Theta_t\|_{\mathcal{L}(\mathcal{H})} = \sup_{\|h\|_{\mathcal{H}} \le 1} \|\Theta_t(h)\|_{\mathcal{H}}$.

Lemma 2.8. *Assume that* $\int_0^\infty \|\Theta_t\|_{\mathcal{L}(\mathcal{H})} \,\mathrm{d}t < 1$. *Then* $\lambda' > 0$ *and so* ν_∞ *is locally stable.*

Proof. Recall that by (7), it holds that $\sup_{t \ge 0} e^{\kappa_* t} \|\Theta_t\|_{\mathcal{L}(\mathcal{H})} < \infty$. Therefore, there exists $\delta > 0$ small enough such that

$$\int_0^\infty e^{\delta t} \|\Theta_t\|_{\mathcal{L}(\mathcal{H})} \,\mathrm{d}t < 1.$$

For $\Re(z) \ge -\delta$, it holds that $\|\hat{\Theta}(z)\|_{\mathcal{L}(\mathcal{H})} \le \int_0^\infty e^{-\Re(z)t} \|\Theta_t\|_{\mathcal{L}(\mathcal{H})} \,\mathrm{d}t < 1$. We deduce that $I - \hat{\Theta}(z)$ is invertible for $\Re(z) \ge -\delta$, with inverse given by $\sum_{k \ge 0} (\hat{\Theta}(z))^k$. So $\lambda' \ge \delta > 0$. □

This assumption is typically satisfied if the non-linear part in (3) is weak enough. Given $M \in M_d(\mathbb{C})$, let $\|M\|_2$ be the spectral norm of the matrix M. We let $[F] := \sup_{x,y \in \mathbb{R}^d} \|\nabla_y F(x, y)\|_2$. Then, from (9), we have $\|\Theta_t\|_{\mathcal{L}(\mathcal{H})} \le C_*[F]e^{-\kappa_* t}$. Therefore, if $[F] < \kappa_*/C_*$, then $\lambda' > 0$, and so any invariant probability measure of (3) is locally stable.

Case of "separable" interactions
Assume that F is "separable", in the sense that there exist functions $w_i : \mathbb{R}^d \to \mathbb{R}^d$ and $f_i : \mathbb{R}^d \to \mathbb{R}$ such that

$$\forall x, y \in \mathbb{R}^d \times \mathbb{R}^d, \quad F(x, y) = \sum_{i=1}^p f_i(y) w_i(x).$$

Let $\mathcal{H}_0$ be the following subspace of $\mathcal{H}$ of finite dimension

$$\mathcal{H}_0 := \left\{ h \mid h = \sum_{i=1}^p \beta_i w_i,\ \beta \in \mathbb{R}^p \right\}.$$

For all $h \in \mathcal{H}$ and for all $t \ge 0$, it holds that $\Theta_t(h) \in \mathcal{H}_0$. The restriction of Θ_t to $\mathcal{H}_0$ can be represented by a $p \times p$ matrix, again denoted by Θ_t, and we have

$$\Theta_t^{i,j} = \int_{\mathbb{R}^d} \nabla_y \mathbb{E}_y f_i\big(Y_t^\alpha\big) \cdot w_j(y) \nu_\infty(\mathrm{d}y), \quad 1 \le i, j \le p.$$

In that case, the determinant in Theorem 2.5 is the standard determinant of matrices. This instance already covers a number of interesting examples.

Case with small noise ($\sigma \simeq 0$)
We now discuss the case where the noise σ is small. The case $\sigma \equiv 0$ would require a special treatment and is not included in Theorem 2.5. It is however instructive to see that the criterion involving (8) is equivalent to the classical stability criterion of a deterministic dynamical systems in $\mathbb{R}^d$. When $\sigma \equiv 0$, the invariant measures are of the form δ_{x_*} for some $x_* \in \mathbb{R}^d$. Let $\mathcal{V}(x, y) = b(x) + F(x, y)$. Assume that

$$\exists \kappa > 0, \quad \sup_{t \ge 0} e^{\kappa t} \big\| e^{t \nabla_x \mathcal{V}(x_*, x_*)} \big\|_2 < \infty. \tag{15}$$

This condition means that x_* is a stable equilibrium point for the ODE $\dot{y} = \mathcal{V}(y, x_*)$. Indeed, (15) is equivalent to the fact that the Jacobian matrix of the vector field $x \mapsto \mathcal{V}(x, x_*)$ has all its eigenvalues in the left half-plane $\Re(z) < 0$.

Lemma 2.9. *Assume* (15) *holds and* $\sigma \equiv 0$. *Then, the criterion* $\lambda' > 0$, λ' *given by* (8), *is equivalent to the fact that the Jacobian matrix of the vector field* $x \mapsto \mathcal{V}(x, x)$ *at the point* x_* *has all its eigenvalues in the left half-plane* $\Re(z) < 0$.

The proof is given in the Appendix. We now treat the case σ small using a perturbation argument of $\sigma \equiv 0$. Assume that for $\|\sigma\|_2$ small enough, the McKean–Vlasov equation (1) has a unique invariant measure in the neighborhood of δ_{x_*}. That is, we assume that there exists $\rho_0, \sigma_0 > 0$ such that for any $\sigma \in M_d(\mathbb{R})$ with $\det(\sigma) > 0$ and $\|\sigma\|_2 \le \sigma_0$, it holds that:

$$\mathrm{Card}\big\{\nu \in \mathcal{P}_1\big(\mathbb{R}^d\big) : W_1(\nu, \delta_{x_*}) \le \rho_0 \text{ and } \nu \text{ is an invariant probability measure of (1)}\big\} = 1.$$

We denote by ν^σ_∞ be this invariant distribution. To emphasize the dependence on σ, until the end of this section, we also denote by Θ^σ_t the operator given by (6) when the SDE (5) has diffusion coefficient σ. Similarly, we denote by Ω^σ_t the operator defined by (11). The precise description of Θ^0_t and Ω^0_t is given in the proof of Lemma 2.9. In particular, we have

$$\Theta^0_t(h)(x) = \nabla_y \mathcal{V}(x, x_*) e^{t\nabla_x \mathcal{V}(x_*,x_*)} \cdot h(x_*).$$

Assume that x_* is a stable point for the ODE $\dot{x} = \mathcal{V}(x, x)$. Then, by (15) and Lemma 2.9, it holds that

$$\exists \kappa > 0, \quad \sup_{t\geq 0} e^{\kappa t}\big[\big\|\Theta^0_t\big\|_{\mathcal{L}(\mathcal{H})} + \big\|\Omega^0_t\big\|_{\mathcal{L}(\mathcal{H})}\big] < \infty. \tag{16}$$

Proposition 2.10. *Assume that* (16) *holds. In addition assume that for all* $\epsilon > 0$, *there exists* $\theta > 0$ *such that*

$$\|\sigma\|_2 \leq \theta \quad \Longrightarrow \quad \sup_{t\geq 0} e^{\kappa t}\big\|\Theta^\sigma_t - \Theta^0_t\big\|_{\mathcal{L}(\mathcal{H})} \leq \epsilon. \tag{17}$$

Then there exists $\sigma_0, \lambda > 0$ *such that*

$$\|\sigma\|_2 \leq \sigma_0, \quad \sup_{t\geq 0} e^{\lambda t}\big\|\Omega^\sigma_t\big\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Therefore Theorem 2.5 *applies and* ν^σ_∞ *is locally stable.*

This result can be summarize has follows: assuming that x_* is a stable equilibrium for the ODE $\dot{x} = \mathcal{V}(x, x)$, if (17) holds then stability also hold with small noise. Again, the proof is given on the Appendix. Finally, let us comment on how the rate of convergence depends on σ. Under the assumptions of Proposition 2.10, the constant λ' given by (8) is lower bounded by a strictly positive constant independent of $\|\sigma\|_2 \in [0, \sigma_0]$. However, the rate of convergence is W_1 norm still depends on σ: the bottleneck is coming from the constant κ_* of (9), which in general vanishes as the noise goes to zero. In some specific examples, the rate of convergence does not vanish as σ goes to zero. Consider for instance the case where $b(x) + \int_{\mathbb{R}^d} F(x, y)\nu^\sigma_\infty(\mathrm{d}y) = -\nabla V^\sigma(x)$ for some function $V^\sigma : \mathbb{R}^d \to \mathbb{R}$. Assume that V^σ is a uniformly convex function, uniformly in $\sigma \in [0, \sigma_0]$:

$$\exists \sigma_0, \eta_0 > 0 : \forall \sigma \text{ with } \|\sigma\|_2 \leq \sigma_0, \forall x \in \mathbb{R}^d, \quad \nabla^2 V^\sigma(x) \geq \eta_0 I_d,$$

where I_d denoted the identity matrix on $M_d(\mathbb{R})$. Then (9) holds with constants independent of σ. In that case, the rate of convergence in W_1 norm in Theorem 2.5 does not vanish as the noise vanishes.

A simple explicit example

We close this Section with a simple explicit example. Consider for $J \in \mathbb{R}^*$ the following McKean–Vlasov SDE on $\mathbb{R}$:

$$\mathrm{d}X_t = -X_t\,\mathrm{d}t + J\mathbb{E}\cos(X_t)\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t. \tag{18}$$

The associated linear process (Y^α_t) is the solution of the Ornstein–Uhlenbeck SDE

$$\mathrm{d}Y^\alpha_t = -Y^\alpha_t\,\mathrm{d}t + \alpha\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t.$$

This linear process admits a unique invariant probability measure given by $\nu^\alpha_\infty = \mathcal{N}(\alpha, 1)$, such that if G is a standard Gaussian random variable, $\mathbb{E}\cos(Y^{\alpha,\nu^\alpha_\infty}_t) = \mathbb{E}\cos(\alpha + G) = \frac{\cos(\alpha)}{\sqrt{e}}$. We deduce that the invariant probability measures of (18) are $\{\mathcal{N}(\alpha, 1) \mid \alpha \in \mathbb{R}, \frac{\sqrt{e}}{J}\alpha = \cos(\alpha)\}$. Let $\alpha \in \mathbb{R}$ such that $\frac{\sqrt{e}}{J}\alpha = \cos(\alpha)$. We have:

$$\forall t \geq 0, \quad \Theta_t = J\int_{\mathbb{R}} \frac{d}{dy}\mathbb{E}_y \cos\big(Y^\alpha_t\big)\nu^\alpha_\infty(\mathrm{d}y) = -\frac{J}{\sqrt{e}}e^{-t}\sin(\alpha).$$

So, for $\Re(z) > -1$, $\widehat{\Theta}(z) = -\frac{J}{z+1}e^{-1/2}\sin(\alpha)$ and the equation $\widehat{\Theta}(z) = 1$ has a unique solution $z = -Je^{-1/2}\sin(\alpha) - 1$. This root is strictly negative if and only if $J\sin(\alpha) > -\sqrt{e}$. We deduce by Theorem 2.5 that ν^α_∞ is locally stable provided that $J\sin(\alpha) > -\sqrt{e}$. Recall that $\alpha\sqrt{e} = J\cos(\alpha)$. So among all the invariant probability measures of (18), the (locally) stable ones are the $\mathcal{N}(\alpha, 1)$ with

$$\alpha\tan(\alpha) > -1.$$

2.3. *Notations*

For $x \in \mathbb{R}^d$, we denote by $|x|$ its Euclidean norm. Recall that $\mathcal{H} := L^2(\nu_\infty)$ is the Hilbert space of measurable functions $h : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\|h\|^2_{\mathcal{H}} := \int \left|h(x)\right|^2 \nu_\infty(\mathrm{d}x) < \infty.$$

We denote by $\mathcal{K} := W^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d)$ the subspace of $\mathcal{H}$ consisting of all bounded and Lipschitz continuous functions $k : \mathbb{R}^d \to \mathbb{R}^d$. We equip $\mathcal{K}$ with:

$$\forall k \in \mathcal{K}, \quad \|k\|_{\mathcal{K}} := \|k\|_\infty + \|\nabla k\|_\infty.$$

Note that, using Assumptions 2.1 and 2.2, the operators Θ_t and Ω_t, defined by (6) and (12), map $\mathcal{H}$ to $\mathcal{K}$. Given I a closed interval of $\mathbb{R}_+$, we denote by $C(I; \mathcal{K})$ the space of continuous functions from I to $\mathcal{K}$. Let $\alpha : \mathbb{R}^d \to \mathbb{R}^d$ satisfying (4). Let $k \in C(\mathbb{R}_+; \mathcal{K})$. Consider $Y^{\alpha+k,\nu}_t$ the solution of the following linear non-homogeneous $\mathbb{R}^d$-valued SDE

$$\mathrm{d}Y^{\alpha+k,\nu}_t = b\big(Y^{\alpha+k,\nu}_t\big)\,\mathrm{d}t + \alpha\big(Y^{\alpha+k,\nu}_t\big)\,\mathrm{d}t + k_t\big(Y^{\alpha+k,\nu}_t\big)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t, \tag{19}$$

where the initial condition satisfies $\mathrm{Law}(Y^{\alpha+k,\nu}_0) = \nu$. Note that $Y^{\alpha+k,\nu}_t$ is a solution of (3) provided that k satisfies the following closure equation:

$$\forall x \in \mathbb{R}^d, \forall t \geq 0, \quad \alpha(x) + k_t(x) = \mathbb{E}F\big(x, Y^{\alpha+k,\nu}_t\big). \tag{20}$$

Finally, for $y \in \mathbb{R}^d$ and g a test function, we write $\mathbb{E}_y g(Y^{\alpha+k}_t) := \mathbb{E}g(Y^{\alpha+k,\delta_y}_t)$.

A key ingredient in the proof of Theorem 2.5 is the notion of Lions derivatives. A function $u : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is Lions differentiable at $\nu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ if there exists a deterministic function $\partial_\nu u(\nu_0) : \mathbb{R}^d \to \mathbb{R}^d$ such that for every random variables Z_0, H defined on the same probabilistic space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathrm{Law}(Z_0) = \nu_0$ and $\mathbb{E}|H|^2 < \infty$, we have for $\nu = \mathrm{Law}(Z_0 + H)$:

$$u(\nu) = u(\nu_0) + \mathbb{E}\big[\partial_\nu u(\mu_0)(Z_0) \cdot H\big] + o\Big(\sqrt{\mathbb{E}|H|^2}\Big), \quad \text{as } \mathbb{E}|H|^2 \text{ goes to zero.}$$

We refer to [10] for a detailed presentation of this theory. Let $\psi \in C^1(\mathbb{R}^d)$ with $\|\nabla\psi\|_\infty < \infty$. We recall that the linear functional

$$u(\nu) = \int_{\mathbb{R}^d} \psi(y)\nu(\mathrm{d}y)$$

is Lions differentiable at every $\nu_0 \in \mathcal{P}_2(\mathbb{R}^d)$, with a Lions derivative given by

$$\partial_\mu u(\nu_0)(y) = \nabla\psi(y).$$

2.4. *An integrated sensibility formula*

Let $\alpha : \mathbb{R}^d \to \mathbb{R}^d$ satisfying (4). The goal of this Section is to prove the following "integrated sensitivity formula".

Proposition 2.11. *Let $k \in C(\mathbb{R}_+; \mathcal{K})$ and $\nu \in \mathcal{P}_1(\mathbb{R}^d)$. Let $g \in C^2(\mathbb{R}^d)$ with $\|\nabla g\|_\infty + \|\nabla^2 g\|_\infty < \infty$. It holds that for all $t \geq 0$,*

$$\mathbb{E}g\big(Y^{\alpha+k,\nu}_t\big) - \mathbb{E}g\big(Y^{\alpha,\nu}_t\big) = \int_0^t \int_{\mathbb{R}^d} \big[\nabla_y \mathbb{E}_y g\big(Y^\alpha_{t-\theta}\big) \cdot k_\theta(y)\big]\mathrm{Law}\big(Y^{\alpha+k,\nu}_\theta\big)(\mathrm{d}y)\,\mathrm{d}\theta.$$

Without loss of generality, we can assume that the initial condition ν belongs to $\mathcal{P}_2(\mathbb{R}^d)$. We define for all $0 \leq \theta \leq t$:

$$\forall \nu \in \mathcal{P}_2\big(\mathbb{R}^d\big), \quad u^g_{t,\theta}(\nu) := \mathbb{E}g\big(Y^{\alpha,\nu}_{t-\theta}\big).$$

By the Markov property, $u^g_{t,\theta}$ is linear with respect to ν:

$$u^g_{t,\theta}(\nu) = \int_{\mathbb{R}^d} \mathbb{E}_y g\big(Y^\alpha_{t-\theta}\big)\nu(\mathrm{d}y).$$

By [29, Th. 7.18], the function $y \mapsto \mathbb{E}_y g(Y^\alpha_{t-\theta})$ is continuously differentiable with a bounded derivative. So $u^g_{t,\theta}$ is Lions differentiable with

$$\partial_\nu u^g_{t,\theta}(\nu)(y) = \nabla_y \mathbb{E}_y g\big(Y^\alpha_{t-\theta}\big).$$

Therefore, it suffices to prove that

$$\mathbb{E}g\big(Y_t^{\alpha+k,\nu}\big) - \mathbb{E}g\big(Y_t^{\alpha,\nu}\big) = \int_0^t \mathbb{E}\big[\partial_\nu u^g_{t,\theta}\big(\mathrm{Law}\big(Y_\theta^{\alpha+k,\nu}\big)\big)\big(Y_\theta^{\alpha+k,\nu}\big) \cdot k_\theta\big(Y_\theta^{\alpha+k,\nu}\big)\big]\,\mathrm{d}\theta.$$

Given $\theta \geq 0$ and $k \in C(\mathbb{R}_+; \mathcal{K})$, we write for all $u \geq 0$:

$$\forall y \in \mathbb{R}^d, \quad k_u^{[\theta]}(y) := \begin{cases} k_u(y) & \text{if } u \leq \theta \\ 0 & \text{if } u > \theta \end{cases} \tag{21}$$

The proof of Proposition 2.11 is deduced from the following Lemma and from the fundamental theorem of calculus.

Lemma 2.12. *The function $\theta \mapsto \mathbb{E}g(Y_t^{\alpha+k^{[\theta]},\nu})$ is differentiable for all $\theta \in (0,t)$ and*

$$\frac{d}{d\theta}\mathbb{E}g\big(Y_t^{\alpha+k^{[\theta]},\nu}\big) = \mathbb{E}\partial_\nu u^g_{t,\theta}\big(\mathrm{Law}\big(Y_\theta^{\alpha+k,\nu}\big)\big)\big(Y_\theta^{\alpha+k,\nu}\big) \cdot k_\theta\big(Y_\theta^{\alpha+k,\nu}\big). \tag{22}$$

Proof. Fix $\theta \in (0,t)$ and $\delta > 0$ small enough such that $\theta + \delta \in (0,t)$. We write

$$Y_\theta := Y_\theta^{\alpha+k,\nu}, \qquad Y^1_{\theta+\delta} := Y^{\alpha+k^{[\theta]},\nu}_{\theta+\delta}, \qquad Y^2_{\theta+\delta} := Y^{\alpha+k,\nu}_{\theta+\delta}$$
$$\mu_\theta := \mathrm{Law}(Y_\theta), \qquad \mu^1_{\theta+\delta} := \mathrm{Law}\big(Y^1_{\theta+\delta}\big), \qquad \mu^2_{\theta+\delta} := \mathrm{Law}\big(Y^2_{\theta+\delta}\big).$$

The notations are illustrated on Figure 1. We have by the Markov property satisfied by Y at time $\theta+\delta$

$$\mathbb{E}g\big(Y_t^{\alpha+k^{[\theta+\delta]},\nu}\big) - \mathbb{E}g\big(Y_t^{\alpha+k^{[\theta]},\nu}\big) = \mathbb{E}g\big(Y^{\alpha,\mu^2_{\theta+\delta}}_{t-(\theta+\delta)}\big) - \mathbb{E}g\big(Y^{\alpha,\mu^1_{\theta+\delta}}_{t-(\theta+\delta)}\big).$$

By definition of the Lions derivative at the point $\mu^1_{\theta+\delta}$ we have

$$\mathbb{E}g\big(Y^{\alpha,\mu^2_{\theta+\delta}}_{t-(\theta+\delta)}\big) - \mathbb{E}g\big(Y^{\alpha,\mu^1_{\theta+\delta}}_{t-(\theta+\delta)}\big) = \mathbb{E}\partial_\nu u^g_{t,\theta+\delta}\big(\mu^1_{\theta+\delta}\big)\big(Y^1_{\theta+\delta}\big)\cdot\big(Y^2_{\theta+\delta} - Y^1_{\theta+\delta}\big) + o\Big(\sqrt{\mathbb{E}\big|Y^2_{\theta+\delta} - Y^1_{\theta+\delta}\big|^2}\Big). \tag{23}$$

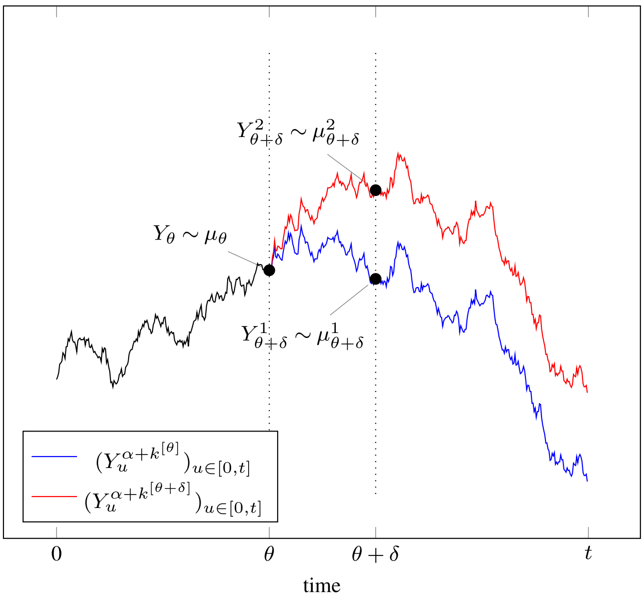

Fig. 1. The two trajectories are driven by the same Brownian motion, with same drift except between times θ and $\theta+\delta$.

By Lemma 2.13 (a) below, it holds that $o(\sqrt{\mathbb{E}|Y^2_{\theta+\delta} - Y^1_{\theta+\delta}|^2}) = o(\delta)$ as δ goes to zero. We now approximate $Y^1_{\theta+\delta}$ and $Y^2_{\theta+\delta}$ by a one-step Euler scheme:

$$\begin{aligned} Y^1_{\theta+\delta} &\approx \tilde{Y}^1_{\theta+\delta} := Y_\theta + \alpha(Y_\theta)\delta + \sigma(W_{\theta+\delta} - W_\theta) \\ Y^2_{\theta+\delta} &\approx \tilde{Y}^2_{\theta+\delta} := Y_\theta + \alpha(Y_\theta)\delta + k_\theta(Y_\theta)\delta + \sigma(W_{\theta+\delta} - W_\theta) \end{aligned} \tag{24}$$

Note that $\tilde{Y}^2_{\theta+\delta} - \tilde{Y}^1_{\theta+\delta} = k_\theta(Y_\theta)\delta$. These one-step Euler schemes have an error in L^2 norm of size $o(\delta)$ (see Lemma 2.13 (c) below), therefore:

$$\sqrt{\mathbb{E}\big|Y^1_{\theta+\delta} - \tilde{Y}^1_{\theta+\delta}\big|^2} + \sqrt{\mathbb{E}\big|Y^2_{\theta+\delta} - \tilde{Y}^2_{\theta+\delta}\big|^2} = o(\delta),$$

so (23) gives

$$\mathbb{E}g\big(Y^{\alpha,\mu^2_{\theta+\delta}}_{t-(\theta+\delta)}\big) - \mathbb{E}g\big(Y^{\alpha,\mu^1_{\theta+\delta}}_{t-(\theta+\delta)}\big) = \delta\mathbb{E}\big[\partial_\nu u^g_{t,\theta+\delta}\big(\mu^1_{\theta+\delta}\big)\big(Y^1_{\theta+\delta}\big)\cdot k_\theta(Y_\theta)\big] + o(\delta).$$

Finally, one has

$$\begin{aligned} &\big|\mathbb{E}\big[\partial_\nu u^g_{t,\theta+\delta}\big(\mu^1_{\theta+\delta}\big)\big(Y^1_{\theta+\delta}\big)\cdot k_\theta(Y_\theta)\big] - \mathbb{E}\big[\partial_\nu u^g_{t,\theta}(\mu_\theta)(Y_\theta)\cdot k_\theta(Y_\theta)\big]\big| \\ &\quad\le \big|\mathbb{E}\big[\partial_\nu u^g_{t,\theta+\delta}\big(\mu^1_{\theta+\delta}\big)\big(Y^1_{\theta+\delta}\big)\cdot k_\theta(Y_\theta)\big] - \mathbb{E}\big[\partial_\nu u^g_{t,\theta+\delta}(\mu_\theta)(Y_\theta)\cdot k_\theta(Y_\theta)\big]\big| \\ &\qquad + \big|\mathbb{E}\big[\partial_\nu u^g_{t,\theta+\delta}(\mu_\theta)(Y_\theta)\cdot k_\theta(Y_\theta) - \mathbb{E}\partial_\nu u^g_{t,\theta}(\mu_\theta)(Y_\theta)\cdot k_\theta(Y_\theta)\big]\big| =: A_1 + A_2. \end{aligned}$$

By Lemma 2.14 (a) there exists a constant $C(t)$ such that

$$A_1 \le C(t)\|k_\theta\|_{\mathcal{K}}\sqrt{\mathbb{E}\big|Y^1_{\theta+\delta} - Y_\theta\big|^2} \overset{\text{Lem. 2.13 }(b)}{\le} C(t)\sqrt{\delta}\sup_{\theta\in[0,t]}\|k_\theta\|_{\mathcal{K}}.$$

Let $\epsilon > 0$ be fixed. Lemma 2.14 (b) yields for δ small enough:

$$A_2 \le \epsilon \sup_{\theta\in[0,t]}\|k_\theta\|_{\mathcal{K}}.$$

Altogether, we find that

$$\mathbb{E}g\big(Y^{a+k^{[\theta+\delta]},\nu}_t\big) - \mathbb{E}g\big(Y^{a+k^{[\theta]},\nu}_t\big) = \delta\mathbb{E}\partial_\nu u^g_{t,\theta}(\mu_\theta)(Y_\theta)\cdot k_\theta(Y_\theta) + o(\delta).$$

This ends the proof. □

We used the following classical estimates (the constants depend on α and k):

Lemma 2.13. *We have, with the notations introduced in the proof of Lemma 2.12,*

1. *it holds that* $\mathbb{E}|Y^2_{\theta+\delta} - Y^1_{\theta+\delta}|^2 \le C(t)\delta^2$.
2. *it holds that* $\mathbb{E}|Y^1_{\theta+\delta} - Y_\theta|^2 \le C(t)\delta$.
3. *the Euler scheme* (24) *satisfies* $\mathbb{E}|Y^1_{\theta+\delta} - \tilde{Y}^1_{\theta+\delta}|^2 + \mathbb{E}|Y^2_{\theta+\delta} - \tilde{Y}^2_{\theta+\delta}|^2 = o(\delta^2)$, *as* δ *goes to zero.*

We also used the following regularity results on $\partial_\nu u^g_{t,s}(\nu)(y) = \nabla_y\mathbb{E}_y g(Y^\alpha_{t-s})$. The proofs follow easily from the stochastic representation of $y \mapsto \nabla_y\mathbb{E}_y g(Y^\alpha_{t-s})$: in particular this function has a bounded derivative (because $\|\nabla g\|_\infty + \|\nabla^2 g\|_\infty < \infty$, see [29, Th. 7.18]).

Lemma 2.14. *It holds that*:

1. *there exists a constant* $C(t)$ *such that any square-integrable variables* Z, Z',

$$\sup_{0\le s\le t}\mathbb{E}\big|\partial_\nu u^g_{t,s}\big(\mathrm{Law}(Z)\big)(Z) - \partial_\nu u^g_{t,s}\big(\mathrm{Law}(Z')\big)(Z')\big|^2 \le C(t)\mathbb{E}\big|Z - Z'\big|^2.$$

2. *the function* $s \mapsto \partial_\nu u^g_{t,s}(\mathrm{Law}(Z))(Z)$ *is continuous*: *for all* $\epsilon > 0$ *there exists* $\delta > 0$:

$$\forall s, s' \in [0,t], \quad |s - s'| \leq \delta \quad \Longrightarrow \quad \mathbb{E}\big|\partial_\nu u^g_{t,s'}\big(\mathrm{Law}(Z)\big)(Z) - \partial_\nu u^g_{t,s}\big(\mathrm{Law}(Z)\big)(Z)\big|^2 < \epsilon.$$

Remark 2.15. It is also possible to prove Proposition 2.11 without using Lions derivatives. Fix $t > 0$ and define for $s \in (0,t)$

$$(s, y) \mapsto \phi(s, y) := \mathbb{E}_y g\big(Y^\alpha_{t-s}\big).$$

Given $k \in C([0,t]; \mathcal{K})$, we denote by

$$\mathcal{L}^{\alpha+k}_\theta \psi := (b + \alpha + k_\theta) \cdot \nabla\psi + \frac{1}{2}\sum_{i,j=1}^d \big(\sigma\sigma^*\big)_{i,j}\partial_{x_i}\partial_{x_j}\psi, \quad \theta \in [0,t],$$

the infinitesimal generator associated to $Y^{\alpha+k}$. It holds that $\phi \in C^{1,2}([0,t) \times \mathbb{R}^d)$ with

$$\frac{\partial}{\partial s}\phi(s,y) = -\mathcal{L}^\alpha_s \phi(s,y).$$

So, by Itô's lemma,

$$\begin{aligned}
\mathbb{E}\phi\big(s, Y^{\alpha+k,\nu}_s\big) &= \mathbb{E}\phi\big(0, Y^{\alpha+k,\nu}_0\big) - \int_0^s \mathbb{E}\mathcal{L}^\alpha_\theta \phi\big(\theta, Y^{\alpha+k,\nu}_\theta\big)\,\mathrm{d}\theta + \int_0^s \mathbb{E}\mathcal{L}^{\alpha+k}_\theta \phi\big(\theta, Y^{\alpha+k,\nu}_\theta\big)\,\mathrm{d}\theta \\
&= \mathbb{E}\phi\big(0, Y^{\alpha+k,\nu}_0\big) + \int_0^s \mathbb{E}\nabla_y\phi\big(\theta, Y^{\alpha+k,\nu}_\theta\big) \cdot k_\theta\big(Y^{\alpha+k,\nu}_\theta\big)\,\mathrm{d}\theta.
\end{aligned}$$

We used that $\mathcal{L}^{\alpha+k}_\theta\psi - \mathcal{L}^\alpha_\theta\psi = \nabla\psi \cdot k_\theta$. Using the definition of ϕ, we find:

$$\mathbb{E}\phi\big(s, Y^{\alpha+k,\nu}_s\big) = \mathbb{E}g\big(Y^{\alpha,\nu}_t\big) + \int_0^s \int_{\mathbb{R}^d} \nabla_y \mathbb{E}_y g\big(Y^\alpha_{t-\theta}\big) \cdot k_\theta(y)\mathcal{L}\big(Y^{\alpha+k,\nu}_\theta\big)(\mathrm{d}y)\,\mathrm{d}\theta.$$

Finally, we let s converges to t and find the stated formula.

As a corollary, we obtain the following apriori control of the Wasserstein distance between two solutions of SDE driven by slightly different drifts. Recall that $C_*, \kappa_* > 0$ are given by (9).

Corollary 2.16. *For all* $t \geq 0$ *and* $k \in C([0,t]; \mathcal{K})$, *it holds that*

$$W_1\big(\mathrm{Law}\big(Y^{\alpha+k,\nu}_t\big), \mathrm{Law}\big(Y^{\alpha,\nu}_t\big)\big) \leq C\int_0^t e^{-\kappa_*(t-\theta)}\|k_\theta\|_\infty\,\mathrm{d}\theta.$$

Proof. Let $g \in C^2(\mathbb{R}^d)$. Using (10) and Proposition 2.11, we have for $g \in C^2(\mathbb{R}^d)$,

$$\big|\mathbb{E}g\big(Y^{\alpha+k,\nu}_t\big) - \mathbb{E}g\big(Y^{\alpha,\nu}_t\big)\big| \leq C_*\|\nabla g\|_\infty \int_0^t e^{-\kappa_*(t-\theta)}\|k_\theta\|_\infty\,\mathrm{d}\theta.$$

This inequality also holds if $g \in C^1(\mathbb{R}^d)$ by a standard approximation argument. □

Note that by choosing $g = F(x, \cdot)$, Proposition 2.11 gives:

$$\mathbb{E}F\big(x, Y^{\alpha+k,\nu}_t\big) - \mathbb{E}F\big(x, Y^{\alpha,\nu}_t\big) = \int_0^t \int_{\mathbb{R}^d} \nabla_y \mathbb{E}_y F\big(x, Y^\alpha_{t-\theta}\big) \cdot k_\theta(y)\mathrm{Law}\big(Y^{\alpha+k,\nu}_\theta\big)(\mathrm{d}y)\,\mathrm{d}\theta. \tag{25}$$

Recall that $\Theta_t(k)$ is defined by (6). When $\nu = \nu_\infty$ and when k is small, we obtain:

$$\mathbb{E}F\big(x, Y^{\alpha+k,\nu_\infty}_t\big) - \alpha(x) \approx \int_0^t \int_{\mathbb{R}^d} \nabla_y \mathbb{E}_y F\big(x, Y^\alpha_{t-\theta}\big) \cdot k_\theta(y)\nu_\infty(\mathrm{d}y)\,\mathrm{d}\theta = \int_0^t \Theta_{t-\theta}(k_\theta)(x)\,\mathrm{d}\theta.$$

This observation is crucially used in the next Section.

2.5. *Control of the non-linear interactions*

Recall that (X_t^ν) denotes the solution of the McKean–Vlasov equation (3). For all $t \geq 0$ and $\nu \in \mathcal{P}_1(\mathbb{R}^d)$, we define

$$\varphi_t^\nu(x) := \mathbb{E}F\big(x, Y_t^{\alpha,\nu}\big) - \alpha(x),$$
$$k_t^\nu(x) := \mathbb{E}F\big(x, X_t^\nu\big) - \alpha(x).$$

Recall that $\Omega_t(h)$ is defined by (11). In this Section, we prove that:

Proposition 2.17. *For all $T > 0$, there is a constant C_T such that for all $t \in [0, T]$, for all $x \in \mathbb{R}^d$ and for all $\nu \in \mathcal{P}_1(\mathbb{R}^d)$:*

$$\left| k_t^\nu(x) - \varphi_t^\nu(x) - \int_0^t \Omega_{t-s}\big(\varphi_s^\nu\big)(x)\,\mathrm{d}s \right| \leq C_T\big(W_1(\nu, \nu_\infty)\big)^2.$$

The first step is to show the following apriori estimate on k_t^ν:

Lemma 2.18. *Let $T > 0$. There exists a constant C_T such that*

$$\forall \nu \in \mathcal{P}_1\big(\mathbb{R}^d\big), \qquad \sup_{t\in[0,T]} \big\|k_t^\nu\big\|_{\mathcal{K}} \leq C_T W_1(\nu, \nu_\infty).$$

In addition, $k^\nu \in C([0, T]; \mathcal{K})$.

Proof. We have

$$k_t^\nu(x) = \int_{\mathbb{R}^d} F(x, y)\big(\mathrm{Law}\big(X_t^\nu\big) - \nu_\infty\big)(\mathrm{d}y).$$

Using Lemma 2.20 below, we deduce that $|k_t^\nu(x)| \leq C_T \|\nabla_y F\|_\infty W_1(\nu, \nu_\infty)$. Similarly,

$$\big|\nabla k_t^\nu(x)\big| \leq C_T \big\|\nabla_{x,y}^2 F\big\|_\infty W_1(\nu, \nu_\infty).$$

This shows the bound on $\sup_{t\in[0,T]}\|k_t^\nu\|_{\mathcal{K}}$. Moreover,

$$k_t^\nu(x) - k_s^\nu(x) = \int_{\mathbb{R}^d} F(x, y)\big(\mathrm{Law}\big(X_t^\nu\big) - \mathrm{Law}\big(X_s^\nu\big)\big)(\mathrm{d}y).$$

In addition, for all $T > 0$ and $\nu \in \mathcal{P}_1(\mathbb{R}^d)$, there exists a constant $C(T, \nu)$ such that

$$\forall 0 \leq s \leq t \leq T, \quad W_1\big(\mathrm{Law}\big(X_t^\nu\big), \mathrm{Law}\big(X_s^\nu\big)\big) \leq \mathbb{E}\big|X_t^\nu - X_s^\nu\big| \leq C(T, \nu)\sqrt{t-s}.$$

We deduce that $k^\nu \in C([0, T]; \mathcal{H})$. □

Next, using Proposition 2.11, we show that:

Lemma 2.19. *For all $T > 0$, there is a constant C_T such that for all $t \in [0, T]$, for all $x \in \mathbb{R}^d$ and for all $\nu \in \mathcal{P}_1(\mathbb{R}^d)$:*

$$\left| k_t^\nu(x) - \varphi_t^\nu(x) - \int_0^t \Theta_{t-s}\big(k_s^\nu\big)(x)\,\mathrm{d}s \right| \leq C_T\big(W_1(\nu, \nu_\infty)\big)^2.$$

Proof. Using the closure equation (20), we have:

$$\begin{aligned} k_t^\nu(x) &= \mathbb{E}F\big(x, Y_t^{\alpha+k^\nu,\nu}\big) - \alpha(x) \\ &= \mathbb{E}F\big(x, Y_t^{\alpha+k^\nu,\nu}\big) - \mathbb{E}F\big(x, Y_t^{\alpha,\nu}\big) + \varphi_t^\nu(x). \end{aligned}$$

We apply Proposition 2.11 and obtain:

$$k_t^\nu(x) - \varphi_t^\nu(x) = \int_0^t \int_{\mathbb{R}^d} \nabla_y \mathbb{E}_y F\big(x, Y_{t-\theta}^\alpha\big) \cdot k_\theta^\nu(y)\mathrm{Law}\big(X_\theta^\nu\big)(\mathrm{d}y)\,\mathrm{d}\theta.$$

We used that $\mathrm{Law}(Y_\theta^{\alpha+k^\nu,\nu}) = \mathrm{Law}(X_\theta^\nu)$. Let

$$G_{t,\theta}^x(y) := \nabla_y \mathbb{E}_y F\big(x, Y_{t-\theta}^\alpha\big) \cdot k_\theta^\nu(y).$$

We deduce that:

$$k_t^\nu(x) - \varphi_t^\nu(x) = \int_0^t \Theta_{t-\theta}\big(k_s^\nu\big)(x)\,\mathrm{d}s + R_t(x),$$

where

$$R_t(x) := \int_0^t \big(\mathbb{E}G_{t,\theta}^x\big(X_\theta^\nu\big) - \mathbb{E}G_{t,\theta}^x\big(X_\theta^{\nu_\infty}\big)\big)\,\mathrm{d}\theta.$$

Using Lemma 2.18, we deduce for all $T > 0$, there exists a constant C_T, such that

$$\forall x \in \mathbb{R}^d, \forall 0 \le \theta \le t \le T, \quad \big\|\nabla G_{t,\theta}^x\big\|_\infty \le C_T W_1(\nu, \nu_\infty).$$

Therefore, using Lemma 2.20 below, we deduce that $|R_t(x)| \le C_T(W_1(\nu, \nu_\infty))^2$. □

By iterating the estimate of Lemma 2.19, we deduce the proof of Proposition 2.17.

Proof of Proposition 2.17. For $i \in \mathbb{N}$, define:

$$\psi_t^i(x) := \varphi_t^\nu(x) + \int_0^t \Theta_{t-\theta}\big(\varphi_\theta^\nu\big)(x)\,\mathrm{d}\theta + \cdots + \int_0^t \Theta_{t-\theta}^{\otimes i}\big(\varphi_\theta^\nu\big)(x)\,\mathrm{d}\theta.$$

Let D_T a constant such that for all $k \in \mathcal{K}$, $\sup_{t\in[0,T]}\|\Theta_t(k)\|_\infty \le D_T\|k\|_\infty$. By induction, we have:

$$\forall t \in [0,T], \quad \big\|\Theta_t^{\otimes i}(k)\big\|_\infty \le (D_T)^i \frac{t^{i-1}}{(i-1)!}\|k\|_\infty.$$

Let C_T be the constant of Lemma 2.19. By induction, we deduce that:

$$\left\|k_t^\nu - \psi_t^i - \int_0^t \Theta_{t-\theta}^{\otimes(i+1)}\big(k_\theta^\nu\big)\,\mathrm{d}\theta\right\|_\infty \le C_T\big(W_1(\nu,\nu_\infty)\big)^2\left(1 + D_T t + \cdots + \frac{(D_T t)^i}{i!}\right).$$

To conclude, it suffices to take the limit $i \to \infty$: ψ_t^i converges uniformly to $\varphi_t^\nu + \int_0^t \Omega_{t-\theta}(\varphi_s^\nu)\,\mathrm{d}s$, and $\int_0^t \Theta_{t-\theta}^{\otimes(i+1)}(k_\theta^\nu)\,\mathrm{d}\theta$ converges uniformly to zero. This ends the proof. □

We used the following apriori estimate on the solution of (3).

Lemma 2.20. *Let $T > 0$. There exists a constant C_T such that for all $\mu_1, \mu_2 \in \mathcal{P}_1(\mathbb{R}^d)$,*

$$\forall t \in [0,T], \quad W_1\big(\mathrm{Law}\big(X_t^{\mu_1}\big), \mathrm{Law}\big(X_t^{\mu_2}\big)\big) \le C_T W_1(\mu_1, \mu_2).$$

Proof. Consider $(X_t^{\mu_1}, X_t^{\mu_2})$ the solutions of (3) coupled with the same Brownian motion. The initial conditions $(X_0^{\mu_1}, X_0^{\mu_2})$ are chosen such that $\mathbb{E}|X_0^{\mu_1} - X_0^{\mu_2}| = W_1(\mu_1, \mu_2)$. Let $\mu_t^1 := \mathrm{Law}(X_t^{\mu_1})$ and $\mu_t^2 := \mathrm{Law}(X_t^{\mu_2})$. From (3) and Assumption 2.1, we have

$$\begin{aligned}\mathbb{E}\big|X_t^{\mu_1} - X_t^{\mu_2}\big| \le{}& \mathbb{E}\big|X_0^{\mu_1} - X_0^{\mu_2}\big| + \mathbb{E}\int_0^t \big|b\big(X_s^{\mu_1}\big) - b\big(X_s^{\mu_2}\big)\big|\,\mathrm{d}s \\ &+ \int_0^t \int_{\mathbb{R}^d} \big|\mathbb{E}F\big(X_s^{\mu_1}, y\big) - \mathbb{E}F\big(X_s^{\mu_2}, y\big)\big|\mathrm{Law}\big(X_s^{\mu_1}\big)(\mathrm{d}y)\,\mathrm{d}s \\ &+ \int_0^t \left|\int_{\mathbb{R}^d} \mathbb{E}F\big(X_s^{\mu_2}, y\big)\big(\mathrm{Law}\big(X_s^{\mu_1}\big) - \mathrm{Law}\big(X_s^{\mu_2}\big)\big)(\mathrm{d}y)\right|\mathrm{d}s.\end{aligned}$$

The functions F and b are Lipschitz, so there exists a constant L such that

$$\mathbb{E}\big|X_t^{\mu_1} - X_t^{\mu_2}\big| \le \mathbb{E}\big|X_0^{\mu_1} - X_0^{\mu_2}\big| + L\int_0^t \mathbb{E}\big|X_s^{\mu_1} - X_s^{\mu_2}\big|\,\mathrm{d}s.$$

By Grönwall's inequality, we deduce that

$$W_1\big(\mathrm{Law}\big(X_t^{\mu_1}\big), \mathrm{Law}\big(X_t^{\mu_2}\big)\big) \le \mathbb{E}\big|X_t^{\mu_1} - X_t^{\mu_2}\big| \le e^{Lt}\mathbb{E}\big|X_0^{\mu_1} - X_0^{\mu_2}\big| = e^{Lt} W_1(\mu_1, \mu_2). \qquad \square$$

2.6. *Proof of Theorem* 2.5

We now give the proof of Theorem 2.5. By combining Proposition 2.17, Corollary 2.16, and Proposition 2.6, we obtain:

Lemma 2.21. *Assume that* $\lambda' > 0$, *where* λ' *is given by* (8). *Let* $\lambda \in (0, \lambda')$. *There exists a constant* C_λ *such that for all* $T > 0$, *there is a constant* C_T *such that for all* $\nu \in \mathcal{P}_1(\mathbb{R}^d)$ *and for all* $t \in [0, T]$:

$$W_1\big(\mathrm{Law}\big(X_t^\nu\big), \nu_\infty\big) \le C_\lambda e^{-\lambda t} W_1(\nu, \nu_\infty) + C_T\big(W_1(\nu, \nu_\infty)\big)^2.$$

Importantly, the constant C_λ above does not depend on T.

Proof. We write

$$\begin{aligned} W_1\big(\mathrm{Law}\big(X_t^\nu\big), \nu_\infty\big) &= W_1\big(\mathrm{Law}\big(Y_t^{\alpha+k^\nu,\nu}\big), \nu_\infty\big) \\ &\le W_1\big(\mathrm{Law}\big(Y_t^{\alpha+k^\nu,\nu}\big), \mathrm{Law}\big(Y_t^{\alpha,\nu}\big)\big) + W_1\big(\mathrm{Law}\big(Y_t^{\alpha,\nu}\big), \nu_\infty\big) \\ &\le C_* \int_0^t e^{-\kappa_*(t-\theta)} \big\|k_\theta^\nu\big\|_\infty \,\mathrm{d}\theta + C_* W_1(\nu, \nu_\infty) e^{-\kappa_* t}. \end{aligned}$$

We used Corollary 2.16 to estimate $W_1(\mathrm{Law}(Y_t^{\alpha+k^\nu,\nu}), \mathrm{Law}(Y_t^{\alpha,\nu}))$ and (9) as well as Markov property to estimate $W_1(\mathrm{Law}(Y_t^{\alpha,\nu}), \nu_\infty)$. Applying Proposition 2.17, we deduce that

$$\int_0^t e^{-\kappa_*(t-\theta)} \big\|k_\theta^\nu\big\|_\infty \,\mathrm{d}\theta \le \int_0^t e^{-\kappa_*(t-\theta)} \left[\big\|\varphi_\theta^\nu\big\|_\infty + \int_0^\theta \big\|\Omega_{\theta-u}\big(\varphi_u^\nu\big)\big\|_\infty \,\mathrm{d}u\right] \mathrm{d}\theta + C_T\big(W_1(\nu, \nu_\infty)\big)^2.$$

The estimate (10) implies that $\|\varphi_\theta^\nu\|_\infty \le C_* W_1(\nu, \nu_\infty)\|\nabla_y F\|_\infty e^{-\kappa_*\theta}$. Fix $\lambda \in (0, \lambda')$. In view of (7), (12) and the proof of Proposition 2.6, the spectral assumption $\lambda' > 0$ implies that there exists a constant C such that

$$\forall h \in \mathcal{H}, \quad \big\|\Omega_t(h)\big\|_\infty \le C e^{-\lambda t}\|h\|_{\mathcal{H}}.$$

Therefore, using that $\lambda < \kappa_*$, we have

$$\int_0^\theta \big\|\Omega_{\theta-u}\big(\varphi_u^\nu\big)\big\|_\infty \,\mathrm{d}u \le C \int_0^\theta e^{-\lambda(\theta-u)} e^{-\kappa_* u} W_1(\nu, \nu_\infty) \,\mathrm{d}u \le C_\lambda e^{-\lambda\theta} W_1(\nu, \nu_\infty).$$

Altogether, we deduce the stated inequality. $\square$

Finally, the proof of Theorem 2.5 is deduced from Lemma 2.21 by following the argument of [7, Proposition 5.2].

Proof of Theorem 2.5. We choose T large enough such that $C_\lambda e^{-\lambda T} \le \frac{1}{4}$. We choose $\epsilon > 0$ small enough such that

$$W_1(\nu, \nu_\infty) \le \epsilon \quad \Longrightarrow \quad C_T\big(W_1(\nu, \nu_\infty)\big)^2 \le \frac{1}{4} W_1(\nu, \nu_\infty).$$

Therefore we have, by induction, provided that $W_1(\nu, \nu_\infty) \le \epsilon$:

$$W_1\big(\mathrm{Law}\big(X_{kT}^\nu\big), \nu_\infty\big) \le (1/2)^k W_1(\nu, \nu_\infty).$$

We write $t = kT + s$ for some $s \in [0, T)$. Using Lemma 2.20, there exists a constant C such that

$$W_1\big(\mathrm{Law}\big(X_t^\nu\big), \nu_\infty\big) \le C(1/2)^k W_1(\nu, \nu_\infty) \le \frac{C}{2} e^{-ct} W_1(\nu, \nu_\infty),$$

where $c := \frac{\log(2)}{T}$. This ends the proof of Theorem 2.5. $\square$

2.7. *Connections with Lions derivatives*

In this Section, we give a probabilistic interpretation of the linear maps $\Theta_t(h)$ and $\Omega_t(h)$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $Z_0, H \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathrm{Law}(Z_0) = \nu_\infty$. We denote by $h : \mathbb{R}^d \to \mathbb{R}^d$ a measurable function such that:

$$\mathbb{P}(\mathrm{d}\omega)\, p.s. \quad \mathbb{E}[H \mid Z_0] = h(Z_0).$$

It follows from $\mathbb{E}|H|^2 < \infty$ and from the Cauchy–Schwarz inequality that $h \in \mathcal{H}$. Define for all $x \in \mathbb{R}^d$, $t \geq 0$ and $\nu \in \mathcal{P}_2(\mathbb{R}^d)$:

$$u_t^x(\nu) := \mathbb{E}F\big(x, Y_t^{\alpha,\nu}\big),$$
$$v_t^x(\nu) := \mathbb{E}F\big(x, X_t^{\nu}\big).$$

Proposition 2.22. *Under Assumptions 2.1 and 2.2, we have*:

1. *There exists a constant C_T such that for all random variables Z_0, H with $\mathrm{Law}(Z_0) = \nu_\infty$ and $\mathbb{E}|H|^2 < \infty$, for all $t \in [0, T]$ and for all x we have, for $\nu = \mathrm{Law}(Z_0 + H)$*:
$$\big|u_t^x(\nu) - \alpha(x) - \Theta_t(h)(x)\big| \leq C_T \mathbb{E}|H|^2.$$
2. *The function u_t^x is Lions differentiable at ν_∞, with a derivative given by*
$$\partial_\nu u_t^x(\nu_\infty)(y) = \nabla_y \mathbb{E}_y F\big(x, Y_t^\alpha\big).$$
3. *There exists a constant C_T such that for all random variables Z_0, H with $\mathrm{Law}(Z_0) = \nu_\infty$ and $\mathbb{E}|H|^2 < \infty$, for all $t \in [0, T]$ and for all x we have, for $\nu = \mathrm{Law}(Z_0 + H)$*:
$$\big|v_t^x(\nu) - \alpha(x) - \Omega_t(h)(x)\big| \leq C_T \mathbb{E}|H|^2.$$
4. *The function v_t^x is Lions differentiable at ν_∞, with a derivative given by*

$$\partial_\nu v_t^x(\nu_\infty)(y) = \nabla_y \mathbb{E}_y F\big(x, Y_t^\alpha\big) + \int_0^t \Omega_{t-s}\big(\nabla_y \mathbb{E}_y F\big(\cdot, Y_s^\alpha\big)\big)(x)\, \mathrm{d}s. \tag{26}$$

Remark 2.23. In particular, from points 1 and 3, we have for all $h \in \mathcal{H}$:

$$\Theta_t(h)(x) = \lim_{\epsilon \to 0} \frac{u_t^x(\mathrm{Law}(Z_0 + \epsilon h(Z_0))) - \alpha(x)}{\epsilon},$$
$$\Omega_t(h)(x) = \lim_{\epsilon \to 0} \frac{v_t^x(\mathrm{Law}(Z_0 + \epsilon h(Z_0))) - \alpha(x)}{\epsilon}.$$

Proof. The first two points follow from the fact that $u_t^x(\nu)$ depends linearly on ν: $u_t^x(\nu) = \mathbb{E}g(Z_0 + H)$, with $g(y) := \mathbb{E}_y F(x, Y_t^\alpha)$. This function g is C^2 and there exists a constant C_T such that for all $t \in [0, T]$, for all $x \in \mathbb{R}^d$, $\|\nabla^2 g\|_\infty \leq C_T$. Therefore:

$$\begin{aligned} u_t^x(\nu) - \alpha(x) &= \mathbb{E}g(Z_0 + H) - \mathbb{E}g(Z_0) = \mathbb{E}\int_0^1 \nabla g(Z_0 + \theta H) \cdot H\, \mathrm{d}\theta \\ &= \mathbb{E}\nabla g(Z_0) \cdot H + \mathbb{E}\int_0^1 \big(\nabla g(Z_0 + \theta H) - \nabla g(Z_0)\big) \cdot H\, \mathrm{d}\theta \\ &= \mathbb{E}\nabla g(Z_0) \cdot H + O_T\big(\mathbb{E}|H|^2\big). \end{aligned}$$

So u_t^x is Lions differentiable at ν_∞ with $\partial_\nu u_t^x(\nu_\infty)(y) = \nabla g(y)$. In addition:

$$\mathbb{E}\nabla g(Z_0) \cdot H = \mathbb{E}\big[\nabla g(Z_0) \cdot \mathbb{E}(H \mid Z_0)\big] = \mathbb{E}\nabla g(Z_0) \cdot h(Z_0) = \Theta_t(h)(x).$$

The third point is a direct consequence of Proposition 2.17 and of the inequality:

$$W_1(\nu, \nu_\infty)^2 \leq W_2(\nu, \nu_\infty)^2 \leq \mathbb{E}|H|^2.$$

To check the last point, it suffices to show that

$$\mathbb{E}\big[\partial_\nu v_t^x(\nu_\infty)(Z_0)\cdot h(Z_0)\big] = \Omega_t(h),$$

where $\partial_\nu v_t^x(\nu_\infty)(y)$ is given by the right-hand side of (26). This equality follows by the linearity of $h \mapsto \Omega_t(h)$ and by Fubini's theorem. □

2.8. *Static bifurcation analysis: A Green–Kubo formula*

Our result also provides some information on the number of invariant probability measures of (3). In this Section, in addition to Assumption 2.1, we assume that F is bounded

$$\sup_{x,y\in\mathbb{R}^d} \big|F(x,y)\big| < \infty,$$

and that the drift is confining in the sense that:

$$\exists R,\beta > 0,\quad |x-y| \ge R \quad\Longrightarrow\quad \big(b(x)-b(y)\big)\cdot(x-y) \le -\beta|x-y|^2.$$

We consider the following open ball of $\mathcal{K}$

$$\mathcal{K}_0 := \big\{\alpha\in\mathcal{K}:\ \|\alpha\|_{\mathcal{K}} \le \|F\|_\infty + \|\nabla_x F\|_\infty\big\}.$$

Let $\alpha\in\mathcal{K}_0$. Then Assumption 2.2 holds with constants R, β independent of α. So the SDE (19) has a unique invariant probability measure, denoted by ν_∞^α. In addition, the bound (9) holds, for some constants C_* and κ_* which do not depend on α. We consider $\Psi:\mathcal{K}_0\to\mathcal{K}_0$, defined by:

$$\forall\alpha\in\mathcal{K}_0,\quad \Psi(\alpha) := x\mapsto \int_{\mathbb{R}^d} F(x,y)\nu_\infty^\alpha(\mathrm{d}y).$$

Note that there is a one-to-one correspondence between the invariant probability measures of (3) and the fixed-points of Ψ in $\mathcal{K}_0$. The following result is a generalization of the Green–Kubo formula [36, Ch. 5]:

Proposition 2.24. *The function Ψ is Frechet differentiable at every $\alpha\in\mathcal{K}_0$ and*

$$D_\alpha\Psi(\alpha)\cdot\epsilon = \int_0^\infty \Theta_t(\epsilon)\,\mathrm{d}t = \hat{\Theta}(\epsilon)(0),\quad \epsilon,\alpha\in\mathcal{K}_0,$$

where Θ_t is given by (6).

Proof. We have for all $T\ge 0$,

$$\begin{aligned}\int_{\mathbb{R}^d} F(x,y)\big(\nu_\infty^{\alpha+\epsilon}-\nu_\infty^\alpha\big)(\mathrm{d}y) &= \big[\mathbb{E}F\big(x,Y_T^{\alpha+\epsilon,\nu_\infty^{\alpha+\epsilon}}\big) - \mathbb{E}F\big(x,Y_T^{\alpha+\epsilon,\nu_\infty^\alpha}\big)\big]\\ &\quad + \big[\mathbb{E}F\big(x,Y_T^{\alpha+\epsilon,\nu_\infty^\alpha}\big) - \mathbb{E}F\big(x,Y_T^{\alpha,\nu_\infty^\alpha}\big)\big]\\ &=: A(x)+B(x).\end{aligned}$$

By (9), there exist positive constants C, κ_* such that $\|A\|_{\mathcal{K}} \le Ce^{-\kappa_* T}W_1(\nu_\infty^{\alpha+\epsilon},\nu_\infty^\alpha)$: this term can be made arbitrarily small by choosing T sufficiently large. In addition, using Proposition 2.11, we have

$$B(x) = \int_0^T\int_{\mathbb{R}^d}\big[\nabla_y\mathbb{E}_yF\big(x,Y_{T-\theta}^\alpha\big)\cdot\epsilon(y)\big]\mathrm{Law}\big(Y_\theta^{\alpha+\epsilon,\nu_\infty^\alpha}\big)(\mathrm{d}y)\,\mathrm{d}\theta.$$

It follows that $\|B\|_{\mathcal{K}} \le C\|\epsilon\|_\infty$. Letting $T\to\infty$ proves that Ψ is continuous. By refining the previous argument, we show that this function is Frechet differentiable with the stated derivative. Define $G_t^x(y) := \nabla_y\mathbb{E}_yF(x,Y_t^\alpha)\cdot\epsilon(y)$. We have $B(x) = \int_0^T \mathbb{E}G_{T-\theta}^x(Y_\theta^{\alpha+\epsilon,\nu_\infty^\alpha})\,\mathrm{d}\theta$ and, by Girsanov's theorem, provided that $\|\epsilon\|_\infty^2 T < 1$, we have:

$$\begin{aligned}\big|\mathbb{E}G_{T-\theta}^x\big(Y_\theta^{\alpha+\epsilon,\nu_\infty^\alpha}\big) - \mathbb{E}G_{T-\theta}^x\big(Y_\theta^{\alpha,\nu_\infty^\alpha}\big)\big| &\le C\big\|G_{T-\theta}^x\big\|_\infty\sqrt{\theta}\|\epsilon\|_\infty\\ &\le Ce^{-\kappa_*(T-\theta)}\|\epsilon\|_\infty^2\sqrt{T}.\end{aligned}$$

Therefore, we deduce that

$$\left| B(x) - \int_0^\infty \int_{\mathbb{R}^d} \nabla_y \mathbb{E}_y F\big(x, Y_\theta^\alpha\big) \cdot \epsilon(y) \nu_\infty^\alpha(\mathrm{d}y)\,\mathrm{d}\theta \right| \le C\big(\sqrt{T}\|\epsilon\|_\infty^2 + e^{-\kappa_* T}\big).$$

Overall, there exists a constant C such that

$$\begin{aligned}&\left| \int_{\mathbb{R}^d} F(x, y)\big(\nu_\infty^{\alpha+\epsilon} - \nu_\infty^\alpha\big)(\mathrm{d}y) - \int_0^\infty \int_{\mathbb{R}^d} \nabla_y \mathbb{E}_y F\big(x, Y_\theta^\alpha\big) \cdot \epsilon(y) \nu_\infty^\alpha(\mathrm{d}y)\,\mathrm{d}\theta \right| \\ &\quad \le C\big[e^{-\kappa_* T} + \|\epsilon\|_\infty^2 \sqrt{T}\big].\end{aligned}$$

We choose $T = 1/\|\epsilon\|_\infty$ and let $\|\epsilon\|_\infty$ goes to zero: the right-hand term is a $o(\|\epsilon\|_\infty)$. The same estimate holds on the derivative of Ψ with respect to x:

$$\begin{aligned}&\left| \int_{\mathbb{R}^d} \nabla_x F(x, y)\big(\nu_\infty^{\alpha+\epsilon} - \nu_\infty^\alpha\big)(\mathrm{d}y) - \int_0^\infty \int_{\mathbb{R}^d} \nabla_y \mathbb{E}_y \nabla_x F\big(x, Y_\theta^\alpha\big) \cdot \epsilon(y) \nu_\infty^\alpha(\mathrm{d}y)\,\mathrm{d}\theta \right| \\ &\quad \le C\big[e^{-\kappa_* T} + \|\epsilon\|_\infty^2 \sqrt{T}\big].\end{aligned}$$

Altogether, we deduce that $\alpha \mapsto \int_{\mathbb{R}^d} F(\cdot, y)\nu_\infty^\alpha(\mathrm{d}y)$ is Frechet differentiable with the stated derivative. □

The interpretation of this result is the following: static bifurcations, leading to a change of the number of invariant probability measures of (3), occur for parameters satisfying $\det(I - \hat{\Theta}(0)) = 0$. On the other hand, we expect Hopf bifurcations to occur at parameters for which $\det(I - \hat{\Theta}(i\omega)) = 0$, for some $\omega > 0$. Altogether, this covers the two canonical ways to break the stability condition $\lambda' > 0$, where λ' is given by (8). The study of these bifurcations is left to future research.

3. McKean–Vlasov of convolution type on the torus

Let $\beta > 0$. We consider the following McKean–Vlasov equation on the torus $\mathbb{T}^d := (\mathbb{R}/2\pi\mathbb{Z})^d$:

$$\mathrm{d}X_t^\nu = -\int_{\mathbb{T}} \nabla W\big(X_t^\nu - y\big)\mu_t(\mathrm{d}y)\,\mathrm{d}t + \sqrt{2\beta^{-1}}\,\mathrm{d}B_t \quad \text{with } \mu_t = \mathrm{Law}\big(X_t^\nu\big) \tag{27}$$

with initial condition $\mathrm{Law}(X_0^\nu) = \nu \in \mathcal{P}(\mathbb{T}^d)$. Here (B_t) is a Brownian motion on $\mathbb{T}^d$. This equation generalizes the Kuramoto model [1,6,28,38], for which $d = 1$ and $W = -\kappa \cos$ for some constant $\kappa \ge 0$. We refer to [11] for a detailed presentation of examples that fit in the framework of (27), as well as a study of the static bifurcations of this equation. In this Section, we study the local stability of the uniform probability measure using the strategy and the tools introduced in Section 2.

3.1. *Main result*

Write the interaction kernel $W : \mathbb{T}^d \to \mathbb{R}$ in Fourier:

$$W(x) = \sum_{n \in \mathbb{Z}^d} \tilde{W}(n) e^{in\cdot x}, \quad x \in \mathbb{T}^d, \tag{28}$$

where $n \cdot x = \sum_{i=1}^d n_i x_i$. Let $|n|^2 = n \cdot n$. The Fourier coefficients of W are given by

$$\tilde{W}(n) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} W(y) e^{-in\cdot y}\,\mathrm{d}y, \quad n \in \mathbb{Z}^d.$$

Assumption 3.1. Assume that $W \in C^3(\mathbb{T}^d)$ and that $\sum_{n\in\mathbb{Z}^d} |n|^2 |\tilde{W}(n)| < \infty$.

The uniform probability measure

$$U(\mathrm{d}x) := \frac{\mathrm{d}x}{(2\pi)^d}$$

is an invariant probability measure of (27) and

Theorem 3.2. *In addition to Assumption 3.1, assume that*

$$\lambda' := \inf_{n\in\mathbb{Z}^d\setminus\{0\}} |n|^2\big(\beta^{-1} + \Re\big(\tilde{W}(n)\big)\big) > 0. \tag{29}$$

Then $U(\mathrm{d}x) = \frac{\mathrm{d}x}{(2\pi)^d}$ is locally stable: there exists $\lambda \in (0,\lambda')$, $\epsilon > 0$ and $C > 1$ such that for all $\nu \in \mathcal{P}(\mathbb{T}^d)$ with $W_1(\nu, U) < \epsilon$, it holds that

$$\forall t \geq 0, \quad W_1\big(\mathrm{Law}\big(X_t^\nu\big), U\big) \leq C W_1(\nu, U) e^{-\lambda t}.$$

Remark 3.3. The constant λ' is the exact analogue of (8) in Section 2. When the interaction kernel W is even, [11] studies the existence of bifurcations of the invariant probability measures of (27), provided that there exists $n \in \mathbb{Z}^d \setminus \{0\}$ such that $\beta^{-1} + \Re(\tilde{W}(n)) = 0$. Therefore, criterion (29) is sharp: we prove that the uniform measure is stable up to the first bifurcation. Note that we do not require here W to be even.

3.2. *Proof*

To simplify the notations, we first assume that $d = 1$. We discuss the case $d > 1$ afterward, most of the arguments being the same. We write $\sigma := \sqrt{2\beta^{-1}}$. The proof is divided into the following steps.

Step 1. Because ∇W is Lipschitz, the equation (27) has a unique path-wise solution satisfying the following apriori estimate:

$$\forall T > 0, \exists C_T : \forall \nu, \mu \in \mathcal{P}(\mathbb{T}), \qquad \sup_{t\in[0,T]} W_1\big(\mathrm{Law}\big(X_t^\nu\big), \mathrm{Law}\big(X_t^\mu\big)\big) \leq C_T W_1(\nu, \mu).$$

Step 2. We define for $\nu \in \mathcal{P}(\mathbb{T})$, $x \in \mathbb{T}$ and $t \geq 0$:

$$k_t^\nu(x) := -\mathbb{E}\nabla W\big(x - X_t^\nu\big).$$

Recall that $U(\mathrm{d}x) = \frac{\mathrm{d}x}{2\pi}$. Because $k_t^U \equiv 0$, we have, by Step 1

$$\big\|k_t^\nu\big\|_\infty = \sup_{x\in\mathbb{T}}\big|k_t^\nu(x) - k_t^U(x)\big| \leq C_T\big\|\nabla^2 W\big\|_\infty W_1(\nu, U).$$

In addition, $x \mapsto k_t^\nu(x)$ is differentiable and

$$\big\|\nabla k_t^\nu\big\|_\infty = \sup_{x\in\mathbb{T}}\big|\nabla k_t^\nu(x) - \nabla k_t^U(x)\big| \leq C_T\big\|\nabla^3 W\big\|_\infty W_1(\nu, U).$$

Step 3. We now use that there exists $C_* > 1$ and $\kappa_* > 0$ such that

$$\forall x, y \in \mathbb{T}, \forall t \geq 0, \quad W_1\big(\mathrm{Law}(x + \sigma B_t), \mathrm{Law}(y + \sigma B_t)\big) \leq C_* e^{-\kappa_* t}|x - y|.$$

We refer to [35, Prop. 4]. We define for all $t \geq 0$, $x \in \mathbb{T}$ and $\nu \in \mathcal{P}(\mathbb{T})$:

$$\varphi_t^\nu(x) := -\mathbb{E}\nabla W\big(x - X_0^\nu - \sigma B_t\big),$$

where X_0^ν is independent of $(B_t)_{t\geq 0}$ and has law ν. Because $\|\nabla^2 W\|_\infty < \infty$, by the preceding result and the dual formulation of the W_1 norm, there exists a constant $C > 0$ such that:

$$\big\|\varphi_t^\nu\big\|_\infty \leq C e^{-\kappa_* t} W_1(\nu, U).$$

Step 4. Let $\mathcal{K} := W^{1,\infty}(\mathbb{T})$ be the space of bounded and Lipschitz continuous functions from $\mathbb{T}$ to $\mathbb{R}$. For $k \in C(\mathbb{R}_+; \mathcal{K})$ and $\nu \in \mathcal{P}(\mathbb{T})$, we consider $(Y_t^{k,\nu})$ the solution of the following linear non-homogeneous SDE

$$\mathrm{d}Y_t^{k,\nu} = k_t\big(Y_t^{k,\nu}\big)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t,$$

starting with $\mathrm{Law}(Y_0^{k,\nu}) = \nu$. Let $g \in C^2(\mathbb{T})$. The integrated sensibility formula of Proposition 2.11 is, in this context:

$$\mathbb{E}g\big(Y_t^{k,\nu}\big) - \mathbb{E}g\big(Y_t^{0,\nu}\big) = \int_0^t \int_{\mathbb{T}} \nabla_y \mathbb{E}_y g(y + \sigma B_{t-\theta}) \cdot k_\theta(y)\mathrm{Law}\big(Y_\theta^{k,\nu}\big)(\mathrm{d}y)\,\mathrm{d}\theta.$$

Step 5. We let for $h \in L^2(\mathbb{T})$ and $x \in \mathbb{T}$:

$$\Theta_t(h)(x) := -\int_{\mathbb{T}} \nabla_y \mathbb{E}_y \nabla W(x - y - \sigma B_t) \cdot h(y) \frac{\mathrm{d}y}{2\pi}.$$

Using that $\mathbb{E}e^{in\sigma B_t} = e^{-\frac{n^2\sigma^2}{2}t} = e^{-\frac{n^2 t}{\beta}}$, we find that the Fourier series of $\Theta_t(h)(x)$ is:

$$\Theta_t(h)(x) = -\sum_{n\in\mathbb{Z}} n^2 \tilde{W}(n)\tilde{h}(n) e^{-\frac{n^2 t}{\beta}} e^{inx}.$$

So Θ_t is diagonal in the Fourier basis $(e^{inx})_{n\in\mathbb{Z}}$ and $\widetilde{\Theta_t(h)}(n) = -n^2\tilde{W}(n)e^{-\frac{n^2 t}{\beta}}\tilde{h}(n)$. In addition, using $|\tilde{h}(n)| \le \|h\|_\infty$ we have:

$$\|\Theta_t(h)\|_\infty \le C_0 e^{-t/\beta}\|h\|_\infty,$$

where $C_0 := \sum_{n\in\mathbb{Z}} n^2|\tilde{W}(n)| < \infty$.

Step 6. We then define $\Omega_t(h)$ to be the unique solution of the Volterra integral equation:

$$\forall t \ge 0, \quad \Omega_t(h) = \Theta_t(h) + \int_0^t \Theta_{t-s}\big(\Omega_s(h)\big)\,\mathrm{d}s.$$

Again, Ω_t is diagonal in the Fourier basis:

$$\Omega_t(h)(x) = -\sum_{n\in\mathbb{Z}} n^2\tilde{W}(n)\exp\big(-n^2 t\big[\beta^{-1} + \tilde{W}(n)\big]\big)\tilde{h}(n)e^{inx}.$$

Let λ' be given by (29). We have:

$$\|\Omega_t(h)\|_\infty \le C_0 e^{-\lambda' t}\|h\|_\infty.$$

So, under the condition $\lambda' > 0$, (Ω_t) decays at an exponential rate towards zero.

Step 7. Let $x \in \mathbb{T}$ be fixed. We now apply Step 4 with $g(y) := -\nabla W(x - y)$, and with $k_t(y) := k_t^\nu(y)$, where k_t^ν is defined in Step 2. Note that with this choice, $Y_t^{k,\nu} = X_t^\nu$ and so $\mathbb{E}g(Y_t^{k,\nu}) = k_t^\nu(x)$. Similarly, $\mathbb{E}g(Y_t^{0,\nu}) = \varphi_t^\nu(x)$, where $\varphi_t^\nu(x)$ is defined in Step 3. Therefore, we have:

$$\begin{aligned} k_t^\nu(x) - \varphi_t^\nu(x) &= \int_0^t \int_{\mathbb{T}} \mathbb{E}\nabla^2 W(x - y - \sigma B_{t-\theta}) \cdot k_\theta^\nu(y)\mathrm{Law}\big(X_\theta^\nu\big)(\mathrm{d}y)\,\mathrm{d}\theta \\ &= \int_0^t \Theta_{t-\theta}\big(k_\theta^\nu\big)(x)\,\mathrm{d}\theta + R_t(x), \end{aligned}$$

where

$$R_t(x) := \int_0^t \mathbb{E}\big[G_{t,\theta}^x\big(X_\theta^\nu\big) - G_{t,\theta}^x\big(X_\theta^U\big)\big]\,\mathrm{d}\theta,$$
$$G_{t,\theta}^x(y) := \mathbb{E}\nabla^2 W(x - y - \sigma B_{t-\theta}) \cdot k_\theta^\nu(y).$$

Using the apriori estimates of Step 2, we deduce that there exists a constant C_T such that for all $0 \le \theta \le t \le T$:

$$\big|\nabla_y G_{t,\theta}^x(y)\big| \le C_T W_1(\nu, U).$$

Using Step 1, we conclude that $|R_t(x)| \le C_T(W_1(\nu, U))^2$. To summarize, we have proven that for all $T > 0$, there exists a constant C_T such that for all $\nu \in \mathcal{P}(\mathbb{T})$ and for all $t \in [0, T]$:

$$\left|k_t^\nu(x) - \varphi_t^\nu(x) - \int_0^t \Theta_{t-\theta}\big(k_s^\nu\big)(x)\,\mathrm{d}\theta\right| \le C_T\big(W_1(\nu, U)\big)^2.$$

Step 8. By iterating the last inequality of Step 7, we obtain that for all $T > 0$, there exists a constant C_T such that

$$\left|k_t^\nu(x) - \varphi_t^\nu(x) - \int_0^t \Omega_{t-\theta}\big(\varphi_s^\nu\big)(x)\,\mathrm{d}\theta\right| \le C_T\big(W_1(\nu, U)\big)^2.$$

Step 9. We prove that there exists a constant $C > 0$ such that for all $t > 0$, for all $k \in C([0,t];\mathcal{K})$ and for all $\nu \in \mathcal{P}(\mathbb{T})$, it holds that

$$W_1\big(\mathrm{Law}\big(Y_t^{k,\nu}\big), \mathrm{Law}\big(Y_t^{0,\nu}\big)\big) \le C \int_0^t e^{-\kappa_*(t-\theta)} \|k_\theta\|_\infty \,\mathrm{d}\theta.$$

The proof is obtained exactly as in Corollary 2.16; it uses the estimates of Step 3.

Step 10. We fix $\lambda \in (0, \min(\kappa_*, \lambda'))$. Using Step 8, Step 6, and Step 3, we deduce that there exists $C_\lambda > 0$ such that for all $T > 0$, there is a constant C_T such that for all $t \in [0,T]$ and $\nu \in \mathcal{P}(\mathbb{T})$:

$$\big\|k_t^\nu\big\|_\infty \le C_T \big(W_1(\nu, U)\big)^2 + C_\lambda W_1(\nu, U) e^{-\lambda t}.$$

Let $k_t(x) := k_t^\nu(x)$. Using that $X_t^\nu = Y_t^{k,\nu}$, we have:

$$W_1\big(\mathrm{Law}\big(X_t^\nu\big), U\big) \le W_1\big(\mathrm{Law}\big(Y_t^{k,\nu}\big), \mathrm{Law}\big(Y_t^{0,\nu}\big)\big) + W_1\big(\mathrm{Law}\big(Y_t^{0,\nu}\big), U\big).$$

By Step 3, we have

$$W_1\big(\mathrm{Law}\big(Y_t^{0,\nu}\big), U\big) \le C_* e^{-\kappa_* t} W_1(\nu, U).$$

By Step 9, we have

$$W_1\big(\mathrm{Law}\big(Y_t^{k,\nu}\big), \mathrm{Law}\big(Y_t^{0,\nu}\big)\big) \le C \int_0^t e^{-\kappa_*(t-\theta)} \big\|k_\theta^\nu\big\|_\infty \,\mathrm{d}\theta.$$

Altogether, we deduce that there is a constant C_λ such that for all $T > 0$, there exists $C_T > 0$ such that for all $t \in [0,T]$, for all $\nu \in \mathcal{P}(\mathbb{T})$, we have:

$$W_1\big(\mathrm{Law}\big(X_t^\nu\big), U\big) \le C_\lambda W_1(\nu, U) e^{-\lambda t} + C_T \big(W_1(\nu, U)\big)^2.$$

The proof of Theorem 3.2 is deduced from this estimate, exactly as we did at the end of Section 2.6. This ends the proof for $d = 1$.

The case $d > 1$ is similar; the only differences are in the expressions of Θ_t and Ω_t of Steps 5 and 6. Given $n \in \mathbb{Z}^d$, we denote by $P_{(n)}$ the $d \times d$ matrix defined by $P_{(n)} = (n_i n_j)_{1 \le i,j \le d}$. We find that for all $h \in L^2(\mathbb{T}^d; \mathbb{R}^d)$ and for all $x \in \mathbb{T}^d$,

$$\Theta_t(h)(x) = -\sum_{n \in \mathbb{Z}^d} e^{in\cdot x} \tilde{W}(n) e^{-\frac{|n|^2 t}{\beta}} P_{(n)} \tilde{h}(n),$$

and

$$\Omega_t(h)(x) = -\sum_{n \in \mathbb{Z}^d} e^{in\cdot x} \tilde{W}(n) e^{-\frac{|n|^2 t}{\beta}} P_{(n)} e^{-t\tilde{W}(n)P_{(n)}} \tilde{h}(n).$$

The eigenvalues of $P_{(n)}$ are $|n|^2$ (of order 1) and zero (of order $d-1$). In addition, it holds that for $\theta \in \mathbb{R}$,

$$\big(e^{\theta P_{(n)}}\big)_{i,j} = \delta_{\{i=j\}} + \frac{n_i n_j}{|n|^2}\big(e^{\theta |n|^2} - 1\big).$$

Therefore, the estimates of Steps 5 and 6 still hold in dimension $d > 1$. This ends the proof.

Appendix

Proof of Lemma 2.9

Recall that $\mathcal{V}(x, y) := b(x) + F(x, y)$. When $\sigma \equiv 0$, then $\nu_\infty = \delta_{x_*}$ for some $x_* \in \mathbb{R}^d$. Therefore (6) writes:

$$\Theta_t(h)(x) = \nabla_y \mathcal{V}(x, x_*) e^{t\nabla_x \mathcal{V}(x_*, x_*)} \cdot h(x_*) =: A_t^x \cdot h(x_*).$$

We look at solutions of (12) of the form: $\Omega_t(h)(x) := B_t^x \cdot h(x_*)$, for some matrices B_t^x. We find that B_t^x satisfies

$$\forall t \ge 0, \quad B_t^x = A_t^x + \int_0^t A_{t-s}^x \cdot B_s^{x_*} \,\mathrm{d}s. \tag{30}$$

We first study this equation for $x = x_*$. For all $z \in \mathbb{C}$ with $\Re(z) > -\kappa$,

$$\widehat{A^{x_*}}(z) = \nabla_y \mathcal{V}(x_*, x_*) \int_0^\infty e^{-t(zI_d - \nabla_x \mathcal{V}(x_*,x_*))}\,\mathrm{d}t = \nabla_y \mathcal{V}(x_*, x_*)\big(zI_d - \nabla_x \mathcal{V}(x_*, x_*)\big)^{-1}.$$

Lemma A.1. *Let $z \in \mathbb{C}$ such that $\Re(z) > -\kappa$. Then $\det(I_d - \widehat{A^{x_*}}(z)) = 0$ if and only if z is an eigenvalue of $\nabla_x \mathcal{V}(x_*, x_*) + \nabla_y \mathcal{V}(x_*, x_*)$.*

Proof. When $\det(I_d - \widehat{A^{x_*}}(z)) = 0$, there exists $u \in \mathbb{R}^d \setminus \{0\}$ such that

$$u = \nabla_y \mathcal{V}(x_*, x_*)\big(zI_d - \nabla_x \mathcal{V}(x_*, x_*)\big)^{-1} u.$$

Setting $v = (zI_d - \nabla_x \mathcal{V}(x_*, x_*))^{-1} u$, we have $v \neq 0$ and

$$zv = \big(\nabla_x \mathcal{V}(x_*, x_*) + \nabla_y \mathcal{V}(x_*, x_*)\big)v.$$

So z is an eigenvalue of $\nabla_x \mathcal{V}(x_*, x_*) + \nabla_y \mathcal{V}(x_*, x_*)$. The converse statement is proved similarly. □

Now, assume that all the real parts of the eigenvalues of $\nabla_x \mathcal{V}(x_*, x_*) + \nabla_y \mathcal{V}(x_*, x_*)$ are less than $-\lambda'$, for some $\lambda' \in (0, \kappa)$. Then, for all $\Re(z) \geq -\lambda'$, $\det(I - A_t^{x_*}) \neq 0$. Applying [30, Th. 4.1], we deduce that $\int_0^\infty e^{\lambda' t} \|B_t^{x_*}\|\,\mathrm{d}t < \infty$. Using (30), one deduces that $\|B_t^{x_*}\| e^{\lambda' t} < \infty$, and so

$$\big\|\Omega_t(h)\big\|_{\mathcal{H}} = \big|\Omega_t(h)(x_*)\big| = \big|B_t^{x_*} \cdot h(x_*)\big| \leq Ce^{-\lambda' t}\big|h(x_*)\big| = Ce^{-\lambda' t}\|h\|_{\mathcal{H}}.$$

Therefore, $\lambda' > 0$. Conversely, if $\lambda' > 0$ where λ' is given by (8) holds, then for $\Re(z) > -\lambda'$, $\det(I_d - \widehat{A^{x_*}}(z)) \neq 0$. So by Lemma A.1, z is not an eigenvalue of $\nabla_x \mathcal{V}(x_*, x_*) + \nabla_y \mathcal{V}(x_*, x_*)$. We deduce that all the eigenvalues of this matrix have real parts less or equal to $-\lambda'$.

Proof of Proposition 2.10

Denote by $\bar{\Theta}_t := \Theta_t^\sigma - \Theta_t^0$ and let $L_t(h) := \bar{\Theta}_t(h) + \int_0^t \Omega_{t-s}^0(\bar{\Theta}_s(h))\,\mathrm{d}s$. We also let R be the resolvent of L, such that R solves

$$R_t = L_t + \int_0^t L_{t-s} \cdot R_s\,\mathrm{d}s.$$

Lemma A.2. *It holds that for all $t \geq 0$, $\Omega_t^\sigma = \Omega_t^0 + R_t + \int_0^t R_{t-s} \cdot \Omega_s^0\,\mathrm{d}s$.*

Proof. Let $Q_t := \Omega_t^0 + R_t + \int_0^t R_{t-s} \cdot \Omega_s^0\,\mathrm{d}s$. To simplify the notation, we write for $A, B \in \mathcal{L}(\mathcal{H})$:

$$(A * B)_t = \int_0^t A_{t-s} \cdot B_s\,\mathrm{d}s.$$

We have $L * Q = L * \Omega^0 + L * R + (L * R * \Omega^0) = R * \Omega^0 + L * R$. Therefore, using that $L * R = R - L$ and the definition of Q, we find that Q solves

$$Q = \Omega^0 + L + L * Q. \tag{31}$$

As $L = \bar{\Theta} + \Omega^0 * \bar{\Theta}$, we have

$$Q - \big(\bar{\Theta} + \Omega^0 * \bar{\Theta}\big) * Q = \Omega^0 + \bar{\Theta} + \Omega^0 * \bar{\Theta}.$$

We multiply on the left by Θ^0. Using that $\Theta^0 * \Omega^0 = \Omega^0 - \Theta^0$, we find that

$$\Theta^0 * Q - \Omega^0 * \bar{\Theta} * Q = \Omega^0 - \Theta^0 + \Omega^0 * \bar{\Theta}.$$

Finally, using that $\bar{\Theta} = \Theta^\sigma - \Theta^0$, we find that $\Theta^0 - \Omega^0 * \bar{\Theta} * Q = \Omega^0 * \Theta^\sigma$. Altogether:

$$L * Q = \Theta^\sigma * Q - \Omega^0 * \Theta^\sigma.$$

We substitute this equality in (31) to finally obtain that

$$Q = \Theta^\sigma + \Theta^\sigma * Q.$$

As Ω^σ is the unique solution of this Volterra integral equation, we deduce that $\Omega^\sigma = Q$ as claimed. □

By Assumptions, there exists $\sigma_0, \kappa > 0$ small enough such that for all $\sigma \in M_d(\mathbb{R})$ with $\det(\sigma) > 0$ and $\|\sigma\|_2 \le \sigma_0$, we have

$$\sup_{t\ge 0} e^{\kappa t} \|L_t\|_{\mathcal{L}(\mathcal{H})} \le \frac{\kappa}{4}.$$

Therefore, we deduce that $\sup_{t\ge 0} e^{\frac{\kappa}{2} t} \|R_t\|_{\mathcal{L}(\mathcal{H})} \le \frac{\kappa}{2}$ and so the stated result follows using Lemma A.2.

Acknowledgments

The author would like to express his gratitude to Etienne Tanré for many suggestions at different steps of this work and hearty encouragement. He also thanks Romain Veltz and René Carmona for their valuable advice. The author would like to thank the anonymous referees, an Associate Editor and the Editor for their constructive comments that improved the quality of this paper.

Funding

This research has received funding from the European Union's Horizon 2020 Framework Programme for Research and Innovation under the Specific Grant Agreement No. 945539 (Human Brain Project SGA3) and was supported by AFOSR FA9550-19-1-0291.

References

[1] J. A. Acebrón, L. L. Bonilla, C. J. Pérez Vicente, F. Ritort and R. Spigler. The Kuramoto model: A simple paradigm for synchronization phenomena. *Rev. Modern Phys.* **77** (2005) 137–185. https://doi.org/10.1103/RevModPhys.77.137

[2] M. Arnaudon and P. Del Moral. A second order analysis of McKean–Vlasov semigroups. *Ann. Appl. Probab.* **30** (2020) 2613–2664. MR4187123 https://doi.org/10.1214/20-AAP1568

[3] S. Benachour, B. Roynette, D. Talay and P. Vallois. Nonlinear self-stabilizing processes. I. Existence, invariant probability, propagation of chaos. *Stochastic Process. Appl.* **75** (1998) 173–201. MR1632193 https://doi.org/10.1016/S0304-4149(98)00018-0

[4] S. Benachour, B. Roynette and P. Vallois. Nonlinear self-stabilizing processes. II. Convergence to invariant probability. *Stochastic Process. Appl.* **75** (1998) 203–224. MR1632197 https://doi.org/10.1016/S0304-4149(98)00019-2

[5] D. Benedetto, E. Caglioti, J. A. Carrillo and M. Pulvirenti. A non-Maxwellian steady distribution for one-dimensional granular media. *J. Stat. Phys.* **91** (1998) 979–990. MR1637274 https://doi.org/10.1023/A:1023032000560

[6] L. Bertini, G. Giacomin and K. Pakdaman. Dynamical aspects of mean field plane rotators and the Kuramoto model. *J. Stat. Phys.* **138** (2010) 270–290. MR2594897 https://doi.org/10.1007/s10955-009-9908-9

[7] L. Bertini, G. Giacomin and C. Poquet. Synchronization and random long time dynamics for mean-field plane rotators. *Probab. Theory Related Fields* **160** (2014) 593–653. MR3278917 https://doi.org/10.1007/s00440-013-0536-6

[8] F. Bolley, J. A. Cañizo and J. A. Carrillo. Stochastic mean-field limit: Non-Lipschitz forces and swarming. *Math. Models Methods Appl. Sci.* **21** (2011) 2179–2210. MR2860672 https://doi.org/10.1142/S0218202511005702

[9] O. A. Butkovsky. On ergodic properties of nonlinear Markov chains and stochastic McKean–Vlasov equations. *Theory Probab. Appl.* **58** (2014) 661–674. MR3403022 https://doi.org/10.1137/S0040585X97986825

[10] R. Carmona and F. Delarue. *Probabilistic Theory of Mean Field Games with Applications. I. Mean Field FBSDEs, Control, and Games*. *Probability Theory and Stochastic Modelling* **83**. Springer, Cham, 2018. MR3752669

[11] J. A. Carrillo, R. S. Gvalani, G. A. Pavliotis and A. Schlichting. Long-time behaviour and phase transitions for the McKean–Vlasov equation on the torus. *Arch. Ration. Mech. Anal.* **235** (2020) 635–690. MR4062483 https://doi.org/10.1007/s00205-019-01430-4

[12] J. A. Carrillo, R. J. McCann and C. Villani. Kinetic equilibration rates for granular media and related equations: Entropy dissipation and mass transportation estimates. *Rev. Mat. Iberoam.* **19** (2003) 971–1018. MR2053570 https://doi.org/10.4171/RMI/376

[13] P. Cattiaux and A. Guillin. Semi log-concave Markov diffusions. In *Séminaire de Probabilités XLVI* 231–292. Springer, Cham, 2014. MR3330820 https://doi.org/10.1007/978-3-319-11970-0_9

[14] F. Coppini. Long time dynamics for interacting oscillators on graphs. *Ann. Appl. Probab.* **32** (2022) 360–391. MR4386530 https://doi.org/10.1214/21-aap1680

[15] Q. Cormier. A mean-field model of Integrate-and-Fire neurons: non-linear stability of the stationary solutions. *Math. Neurosci. Appl.* **4** (2024) 1–24. https://doi.org/10.46298/mna.12583

[16] Q. Cormier, E. Tanré and R. Veltz. Long time behavior of a mean-field model of interacting neurons. *Stochastic Process. Appl.* **130** (2020) 2553–2595. MR4080722 https://doi.org/10.1016/j.spa.2019.07.010

[17] Q. Cormier, E. Tanré and R. Veltz. Hopf bifurcation in a mean-field model of spiking neurons. *Electron. J. Probab.* **26** (2021) 121, 40. MR4316639 https://doi.org/10.1214/21-ejp688

[18] P. Degond, A. Frouvelle and J.-G. Liu. Phase transitions, hysteresis, and hyperbolicity for self-organized alignment dynamics. *Arch. Ration. Mech. Anal.* **216** (2015) 63–115. MR3305654 https://doi.org/10.1007/s00205-014-0800-7

[19] P. Del Moral and J. Tugaut. Uniform propagation of chaos and creation of chaos for a class of nonlinear diffusions. *Stoch. Anal. Appl.* **37** (2019) 909–935. MR4020054 https://doi.org/10.1080/07362994.2019.1622426

[20] F. Delarue, J. Inglis, S. Rubenthaler and E. Tanré. Global solvability of a networked integrate-and-fire model of McKean–Vlasov type. *Ann. Appl. Probab.* **25** (2015) 2096–2133. MR3349003 https://doi.org/10.1214/14-AAP1044

[21] F. Delarue and A. Tse. Uniform in time weak propagation of chaos on the torus, 2021.

[22] R. L. Dobrušin. Vlasov equations. *Funktsional. Anal. i Prilozhen.* **13** (1979) 48–58, 96. MR0541637

[23] A. Eberle. Reflection couplings and contraction rates for diffusions. *Probab. Theory Related Fields* **166** (2016) 851–886. MR3568041 https://doi.org/10.1007/s00440-015-0673-1

[24] A. Eberle, A. Guillin and R. Zimmer. Quantitative Harris-type theorems for diffusions and McKean–Vlasov processes. *Trans. Amer. Math. Soc.* **371** (2019) 7135–7173. MR3939573 https://doi.org/10.1090/tran/7576

[25] N. Fournier, M. Hauray and S. Mischler. Propagation of chaos for the 2D viscous vortex model. *J. Eur. Math. Soc. (JEMS)* **16** (2014) 1423–1466. MR3254330 https://doi.org/10.4171/JEMS/465

[26] N. Fournier and E. Löcherbach. On a toy model of interacting neurons. *Ann. Inst. Henri Poincaré Probab. Stat.* **52** (2016) 1844–1876. MR3573298 https://doi.org/10.1214/15-AIHP701

[27] A. Ganz Bustos. Approximations des distributions d'équilibre de certains systèmes stochastiques avec interactions McKean–Vlasov. PhD thesis, Université de Nice, 2008.

[28] G. Giacomin, K. Pakdaman and X. Pellegrin. Global attractor and asymptotic dynamics in the Kuramoto model for coupled noisy phase oscillators. *Nonlinearity* **25** (2012) 1247–1273. MR2914138 https://doi.org/10.1088/0951-7715/25/5/1247

[29] C. Graham and D. Talay. *Stochastic Simulation and Monte Carlo Methods: Mathematical Foundations of Stochastic Simulation*. *Stochastic Modelling and Applied Probability* **68**. Springer, Heidelberg, 2013. MR3097957 https://doi.org/10.1007/978-3-642-39363-1

[30] G. Gripenberg, S. O. Londen and O. Staffans. *Volterra Integral and Functional Equations*. *Encyclopedia of Mathematics and Its Applications* **34**. Cambridge University Press, Cambridge, 1990. MR1050319 https://doi.org/10.1017/CBO9780511662805

[31] A. Guillin and P. Monmarché. Uniform long-time and propagation of chaos estimates for mean field kinetic particles in non-convex landscapes. *J. Stat. Phys.* **185** (2021) 15, 20. MR4333408 https://doi.org/10.1007/s10955-021-02839-6

[32] M. Hauray and P.-E. Jabin. N-Particles approximation of the Vlasov equations with singular potential. *Arch. Ration. Mech. Anal.* **183** (2007) 489–524. MR2278413 https://doi.org/10.1007/s00205-006-0021-9

[33] D. Henry. *Geometric Theory of Semilinear Parabolic Equations*. *Lecture Notes in Mathematics* **840**. Springer-Verlag, Berlin–New York, 1981. MR0610244

[34] S. Herrmann and J. Tugaut. Non-uniqueness of stationary measures for self-stabilizing processes. *Stochastic Process. Appl.* **120** (2010) 1215–1246. MR2639745 https://doi.org/10.1016/j.spa.2010.03.009

[35] L. Journel and P. Monmarché. Convergence of a particle approximation for the quasi-stationary distribution of a diffusion process: Uniform estimates in a compact soft case. *ESAIM Probab. Stat.* **26** (2022) 1–25. MR4363453 https://doi.org/10.1051/ps/2021017

[36] T. Lelièvre and G. Stoltz. Partial differential equations and stochastic methods in molecular dynamics. *Acta Numer.* **25** (2016) 681–880. MR3509213 https://doi.org/10.1017/S0962492916000039

[37] E. Löcherbach and P. Monmarché. Metastability for systems of interacting neurons. *Ann. Inst. Henri Poincaré Probab. Stat.* **58** (2022) 343–378. MR4374679 https://doi.org/10.1214/21-aihp1164

[38] E. Luçon and C. Poquet. Long time dynamics and disorder-induced traveling waves in the stochastic Kuramoto model. *Ann. Inst. Henri Poincaré Probab. Stat.* **53** (2017) 1196–1240. MR3689966 https://doi.org/10.1214/16-AIHP753

[39] E. Luçon and C. Poquet. Periodicity induced by noise and interaction in the kinetic mean-field FitzHugh–Nagumo model. *Ann. Appl. Probab.* **31** (2021) 561–593. MR4254489 https://doi.org/10.1214/20-aap1598

[40] T. Ma and S. Wang. *Bifurcation Theory and Applications*. *World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises* **53**. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005. MR2310258 https://doi.org/10.1142/9789812701152

[41] F. Malrieu. Convergence to equilibrium for granular media equations and their Euler schemes. *Ann. Appl. Probab.* **13** (2003) 540–560. MR1970276 https://doi.org/10.1214/aoap/1050689593

[42] M. Reed and B. Simon *Methods of Modern Mathematical Physics. I: Functional Analysis. Rev. and Enl. Ed*. Academic Press, A Subsidiary of Harcourt Brace Jovanovich, New York, 1980. XV, 400 p. $ 24.00. MR0751959

[43] M. Scheutzow. Some examples of nonlinear diffusion processes having a time-periodic law. *Ann. Probab.* **13** (1985) 379–384. MR0781411

[44] B. Simon. Notes on infinite determinants of Hilbert space operators. *Adv. Math.* **24** (1977) 244–273. MR0482328 https://doi.org/10.1016/0001-8708(77)90057-3

[45] A.-S. Sznitman. Topics in propagation of chaos. In *École d'Été de Probabilités de Saint-Flour XIX – 1989* 165–251. *Lecture Notes in Math.* **1464**. Springer, Berlin, 1991. MR1108185 https://doi.org/10.1007/BFb0085169

[46] D. Talay. Second-order discretization schemes of stochastic differential systems for the computation of the invariant law. *Stoch. Stoch. Rep.* **29** (1990) 13–36. https://doi.org/10.1080/17442509008833606

[47] Y. Tamura. On asymptotic behaviors of the solution of a nonlinear diffusion equation. *J. Fac. Sci., Univ. Tokyo, Sect. 1A, Math.* **31** (1984) 195–221. MR0743525

[48] J. Tugaut. Convergence to the equilibria for self-stabilizing processes in double-well landscape. *Ann. Probab.* **41** (2013) 1427–1460. MR3098681 https://doi.org/10.1214/12-AOP749

[49] J. Tugaut. Phase transitions of McKean–Vlasov processes in double-wells landscape. *Stochastics* **86** (2014) 257–284. MR3180036 https://doi.org/10.1080/17442508.2013.775287

[50] A. Y. Veretennikov. On ergodic measures for McKean–Vlasov stochastic equations. In *Monte Carlo and Quasi-Monte Carlo Methods 2004* 471–486. Springer, Berlin, 2006. MR2208726 https://doi.org/10.1007/3-540-31186-6_29

[51] G. Wolansky. On nonlinear stability of polytropic galaxies. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **16** (1999) 15–48. MR1668556 https://doi.org/10.1016/S0294-1449(99)80007-9